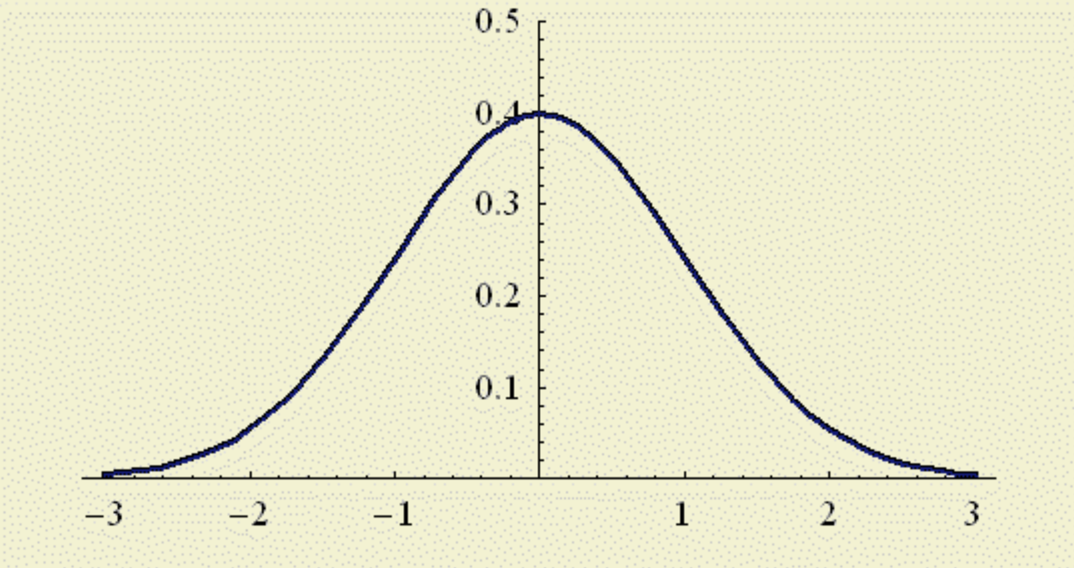
The "Second Law" of Probability: Entropy Growth in the Central Limit Theorem.

Keith Ball



The second law of thermodynamics

Joule and Carnot studied ways to improve the efficiency of steam engines.

Is it possible for a thermodynamic system to move from state A to state B without any net energy being put into the system from outside?

A single experimental quantity, dubbed **entropy**, made it possible to decide the direction of thermodynamic changes.

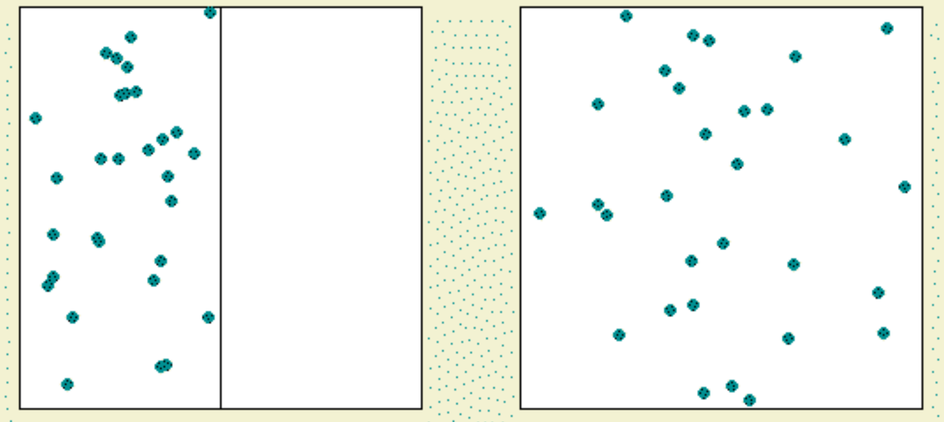
The second law of thermodynamics

The entropy of a closed system increases with time.

The second law applies to all changes: not just thermodynamic.

Entropy measures the extent to which energy is dispersed: so the second law states that energy tends to disperse.

The second law of thermodynamics



Closed systems become progressively more featureless.

We expect that a closed system will approach an equilibrium with maximum entropy.

Information theory

Shannon showed that a noisy channel can communicate information with almost perfect accuracy, up to a fixed rate: the capacity of the channel.

The (Shannon) entropy of a probability distribution: if the possible states have probabilities $p_1, p_2, \ldots, p_n$ then the entropy is

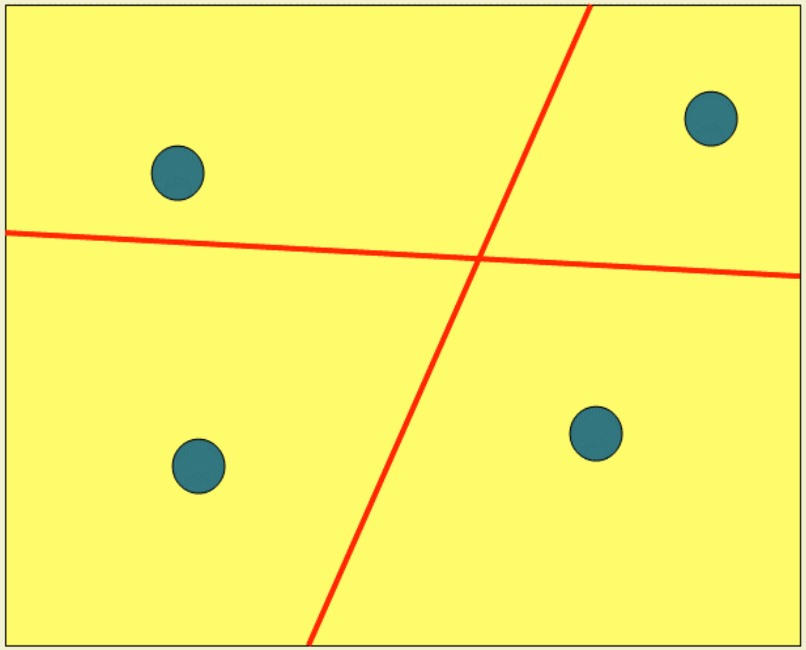
$$! \qquad p_i \log_2 p_i$$

Entropy measures the number of (YES/NO) questions that you expect to have to ask in order to find out which state has occurred.

Information theory

You can distinguish 2^k states with k (YES/NO) questions.

If the states are equally likely, then this is the best you can do.



It costs k questions to identify a state from among 2^k equally likely states.

Information theory

It costs k questions to identify a state from among 2^k equally likely states.

It costs $\log_2 n$ questions to identify a state from among n equally likely states: to identify a state with probability $1/n$.

Probability	Questions
$1/n$	$\log_2 n$
p	$\log_2 (1/p)$

The entropy

State	Probability	Questions	Uncertainty
S_1	p_1	$\log_2 (1/p_1)$	$p_1 \log_2 (1/p_1)$
S_2	p_2	$\log_2 (1/p_2)$	$p_2 \log_2 (1/p_2)$
S_3	p_3	$\log_2 (1/p_3)$	$p_3 \log_2 (1/p_3)$

Entropy $= p_1 \log_2 (1/p_1) + p_2 \log_2 (1/p_2) + p_3 \log_2 (1/p_3) + \ldots$

Continuous random variables

For a random variable X with density f the entropy is

$$\mathrm{Ent}(X) = ! \qquad f \log f$$

The entropy behaves nicely under several natural processes: for example, the evolution governed by the heat equation.

If the density f measures the distribution of heat in an infinite metal bar, then f evolves according to the heat equation:

$$\frac{\partial f}{\partial t} = f''$$

The entropy increases:

$$\frac{\partial}{\partial t}\Big(\quad f \log f \Big) = \underbrace{\quad \frac{f'^2}{f}}_{\textbf{Fisher information}} \geq 0$$

The central limit theorem

If X_i are independent copies of a random variable with mean 0 and finite variance, then the normalized sums

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

converge to a Gaussian (normal) with the same variance.

Most proofs give little intuition as to why.

The central limit theorem

Among random variables with a given variance, the Gaussian has largest entropy.

Theorem (Shannon-Stam) If X and Y are independent and identically distributed, then the normalized sum

$$\frac{X+Y}{\sqrt{2}}$$

has entropy at least that of X and Y.

Idea

The central limit theorem is analogous to the second law of thermodynamics: the normalized sums

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

have increasing entropy which drives them to an "equilibrium" which has maximum entropy.

Problem: (folklore or Lieb (1978)).

Is it true that $\mathrm{Ent}(S_n)$ increases with n?

Shannon-Stam shows that it increases as n goes from 1 to 2 (hence 2 to 4 and so on). Carlen and Soffer found uniform estimates for entropy jump from 1 to 2.

It wasn’t known that entropy increases from 2 to 3.

The difficulty is that you can't express the sum of 3 independent random variables in terms of the sum of 2: you can't add 3/2 independent copies of X.

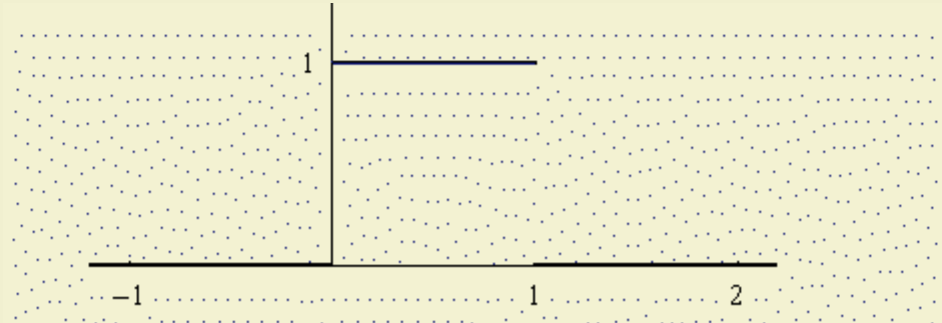
The Fourier transform?

The simplest proof (conceptually) of the central limit theorem uses the FT. If X has density f whose FT is $!$ then the FT of the density of $! \; X_i$ is $!^n$.

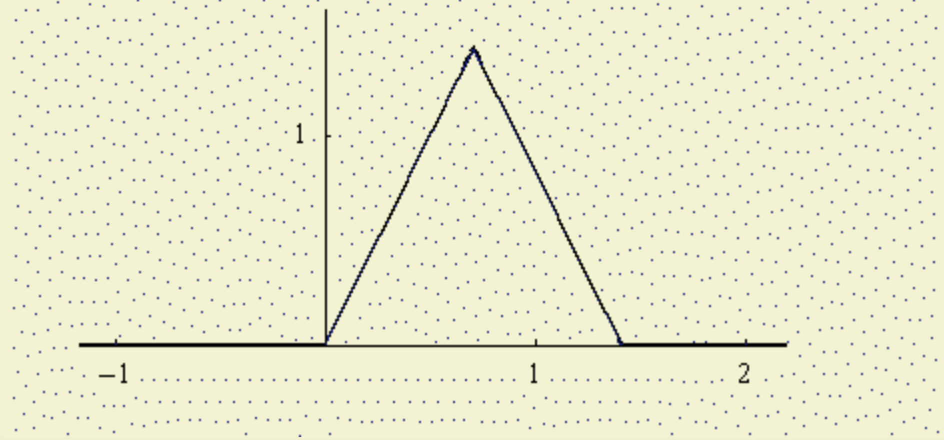
The problem is that the entropy cannot easily be expressed in terms of the FT. So we must stay in real space instead of Fourier space.

Example:

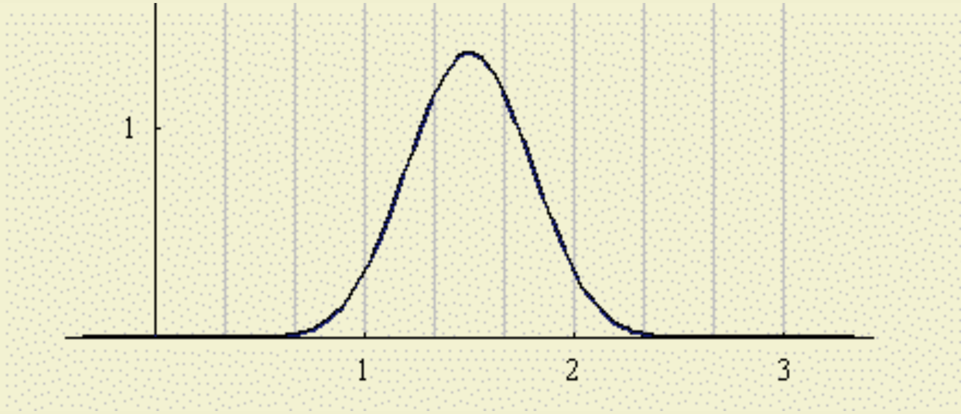
Suppose X is uniformly distributed on the interval between 0 and 1. Its density is:



When we add two copies the density is:



For 9 copies the density is a spline defined by 9 different polynomials on different parts of the range.



The central polynomial (for example) is:

$$3\left(857291 - 5027400\,x + 12800340\,x^2 - 18438840\,x^3 + 16391970\,x^4 - 9185400\,x^5 + 3163860\,x^6 - 612360\,x^7 + 51030\,x^8\right)/4480$$

and its logarithm is?

The second law of probability

A new variational approach to entropy gives quantitative measures of entropy growth and proves the "second law".

Theorem (Artstein, Ball, Barthe, Naor) If X_i are independent copies of a random variable with finite variance, then the normalized sums

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

have increasing entropy.

Starting point: used by many authors. Instead of considering entropy directly, we study the Fisher information:

$$J(X) = \int \frac{f'^2}{f}$$

Among random variables with variance 1, the Gaussian has the **smallest** Fisher information, namely 1.

The Fisher information should decrease as a process evolves.

The connection (we want) between entropy and Fisher information is provided by the Ornstein-Uhlenbeck process (de Bruijn, Bakry and Emery, Barron).

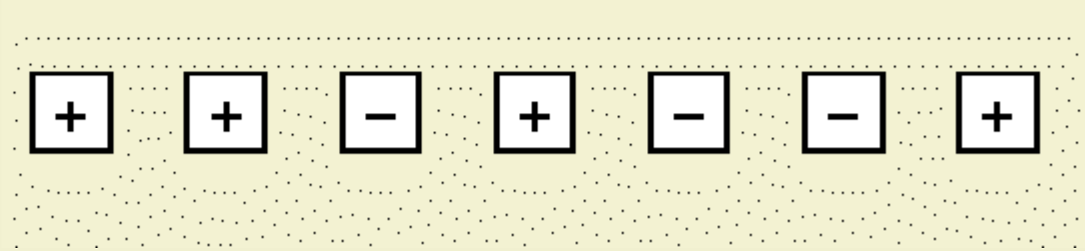
Recall that if the density of $X^{(t)}$ evolves according to the heat equation then

$$\frac{\partial}{\partial t}\mathrm{Ent}(X^{(t)}) = J(X^{(t)})$$

The heat equation can be solved by running a Brownian motion from the initial distribution. The Ornstein-Uhlenbeck process is like Brownian motion but run in a potential which keeps the variance constant.

The Ornstein-Uhlenbeck process

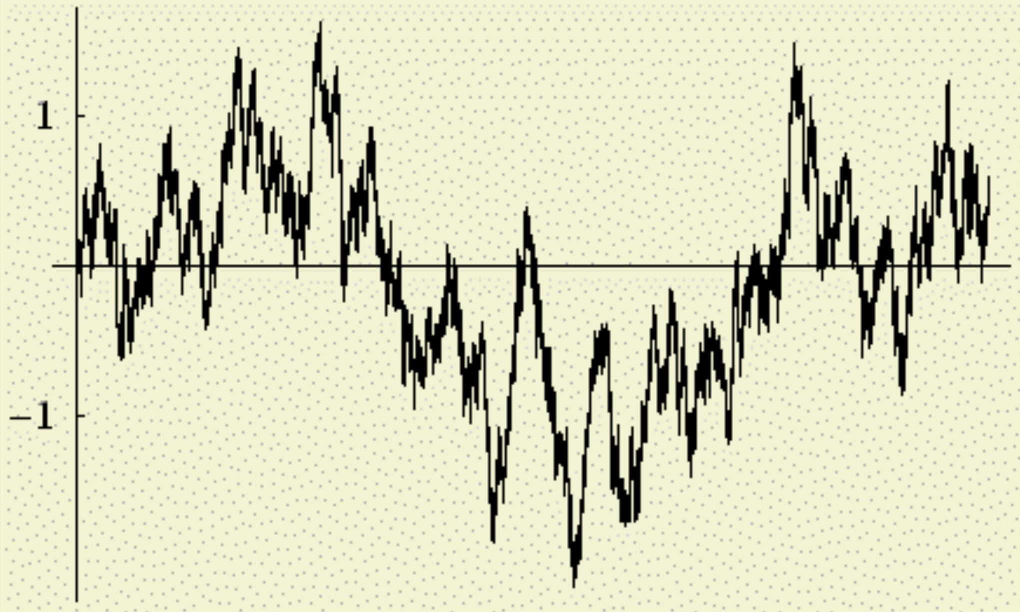
A discrete analogue:



You have n sites, each of which can be ON or OFF. At each time, pick a site (uniformly) at random and switch it.

$$X^{(t)} = \text{(number on)-(number off)}.$$

The Ornstein-Uhlenbeck process



A typical path of the process.

The Ornstein-Uhlenbeck evolution

The density evolves according to the modified diffusion equation:

$$\frac{\partial f}{\partial t} = f'' + (xf)'$$

From this:

$$\frac{\partial}{\partial t}\mathrm{Ent}(X^{(t)}) = J(X^{(t)}) \quad 1$$

As $t \to$ the evolutes approach the Gaussian of the same variance.

The entropy gap can be found by integrating the information gap along the evolution.

$$\mathrm{Ent}(G) - \mathrm{Ent}(X^{(0)}) = \int_0^\infty \left(J(X^{(t)}) - 1 \right)$$

In order to prove entropy increase, it suffices to prove that the information

$$J(n) = J\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right)$$

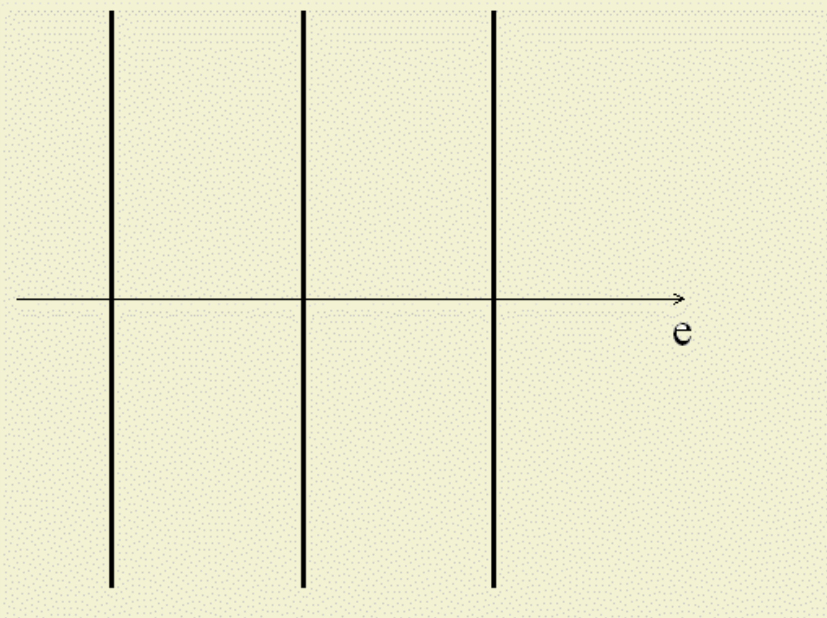
decreases with *n*.

It was known (Blachman-Stam) that $J(2) \le J(1)$.

Main new tool: a variational description of the information of a marginal density.

If w is a density on ^{n} and e is a unit vector, then the marginal in direction e has density

$$h(t) = \quad {}_{te+\langle e\rangle^{!}} \; w$$



Main new tool:

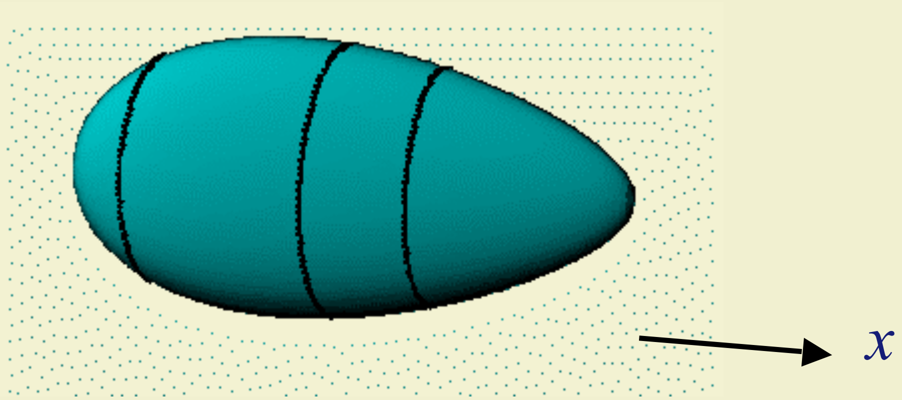
The density h is a marginal of w and

$$J(h) = \frac{h'(t)^2}{h(t)} = \frac{h'(t)^2}{h(t)} \,!\, h''(t) = h(!\,\log h)''$$

The integrand is non-negative if h has concave logarithm.

Densities with concave logarithm have been widely studied in high-dimensional geometry, because they naturally generalize convex solids.

The Brunn-Minkowski inequality

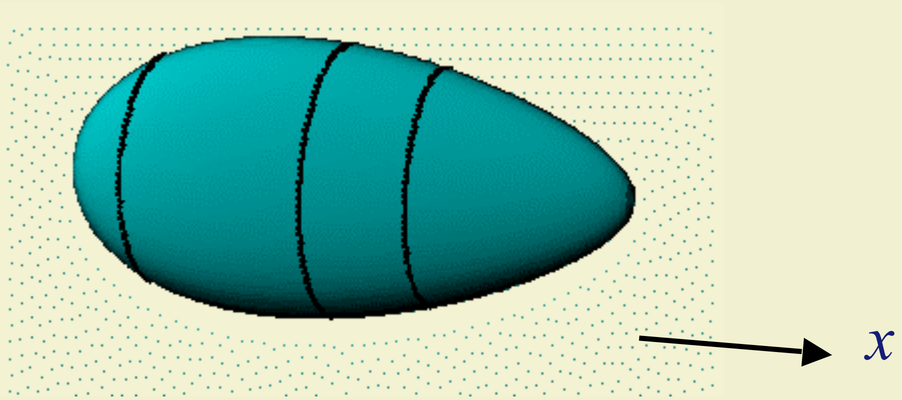


Let $A(x)$ be the cross-sectional area of a convex body at position x.

Then log A is concave.

The function A is a marginal of the body.

The Brunn-Minkowski inequality



We can replace the body by a function with concave logarithm. If w has concave logarithm, then so does each of its marginals.

If the density h is a marginal of w, the inequality tells us something about $(-\log h)''$ in terms of $\mathrm{Hess}(-\log w)$

The Brunn-Minkowski inequality

If the density h is a marginal of w, the inequality tells us something about $(-\log h)''$ in terms of $\mathrm{Hess}(-\log w)$

We rewrite a proof of the Brunn-Minkowski inequality so as to provide an explicit relationship between the two. The expression involving the Hessian is a quadratic form whose minimum is the information of h.

This gives rise to the variational principle.

The variational principle

Theorem If w is a density and e a unit vector then the information of the marginal in the direction e is

$$J(h) = ! \frac{h'(t)^2}{h(t)} dt = \min ! _{n} \frac{\mathrm{div}(pw)^2}{w}$$

where the minimum is taken over vector fields p satisfying

$! p, e \;\# 1.$

$$J(h) = \int \frac{h'(t)^2}{h(t)} dt = \min \int_n \frac{\operatorname{div}(pw)^2}{w}$$

Technically we have gained because $h(t)$ is an integral: not good in the denominator.

The real point is that we get to choose p. Instead of choosing the optimal p which yields the intractable formula for information, we choose a non-optimal p with which we can work.

Proof of the variational principle.

$$h(t) = \quad {}_{te+\langle e\rangle^{!}} \; w$$

so

$$h'(t) = \#_{te+\langle e\rangle^{!}} \quad {}_{e}w$$

If p satisfies $!\, p, e \;\; \# \, 1$ at each point, then we can realise the derivative as

$$h'(t) = \quad {}_{te+\langle e\rangle^{!}} \; \mathrm{div}(pw)$$

since the part of the divergence perpendicular to e integrates to 0 by the Gauss-Green (divergence) theorem.

Hence

$$\int \frac{h'(t)^2}{h(t)}\,dt = \int \frac{\left(\int_{te+\ e\#!} \mathrm{div}(pw)\right)^2}{\int w}\,dt \quad \int_{n} \frac{\mathrm{div}(pw)^2}{w}$$

There is equality if

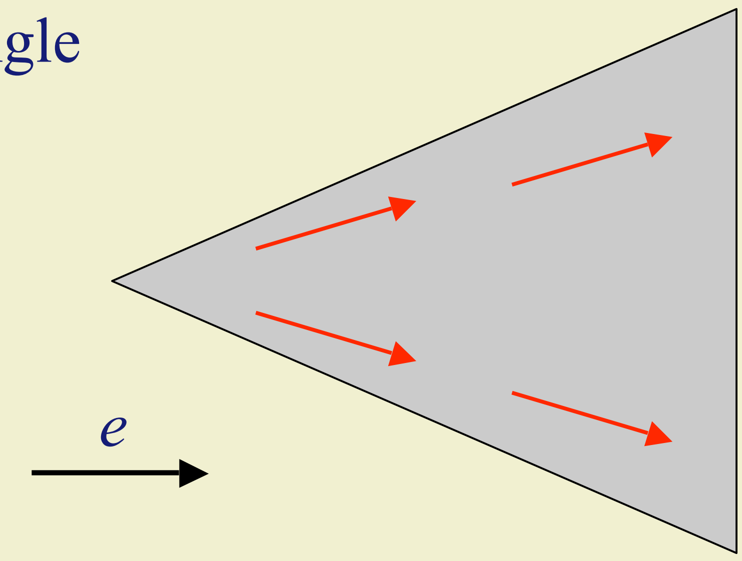
$$\mathrm{div}(pw) = \frac{h'(t)}{h(t)}\,w$$

This divergence equation has many solutions: for example we might try the electrostatic field solution. But this does not decay fast enough at infinity to make the divergence theorem valid.

$$\mathrm{div}(pw) = \frac{h'(t)}{h(t)} w$$

The right solution for p is a flow in the direction of e which transports between the probability measures induced by w on hyperplanes perpendicular to e.

For example, if w is 1 on a triangle and 0 elsewhere, the flow is as shown. (The flow is irrelevant where $w = 0$.)



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