Use Level Sets And Relax!

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Shape optimization.

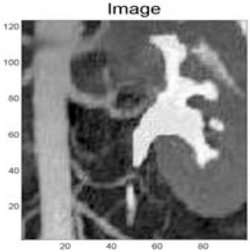
A rather general shape optimization problem.

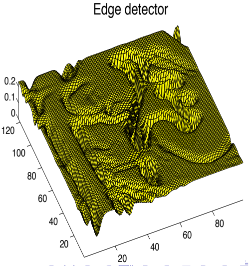
$$\boxed{\text{minimize} \quad f(\mathcal{D}) \quad \text{over a set of feasible geometries } \mathcal{D}}$$

Possible scenarios:

- $f(\mathcal{D})$ per se; e.g., edge detector based image segmentation with $\mathcal{D} = \Gamma = \partial\Omega$ and g_I an edge detector,

$$f(\Gamma) = \int_\Gamma g_I \, dS + \nu \int_\Omega g_I \, d\mathbf{x}.$$

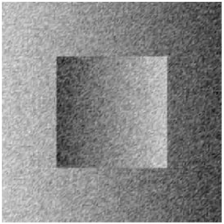


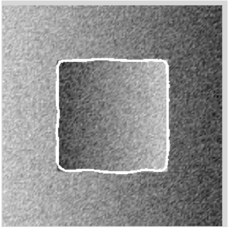


Shape optimization.

- $f(\mathcal{D}) = J(u(\mathcal{D}), \mathcal{D})$ where $u(\mathcal{D})$ solves PDE depending on $\mathcal{D}$; e.g., reduction approach to Mumford-Shah model: $\mathcal{D} = \Gamma$, $f(\mathcal{D}) = J(u(\Gamma), \Gamma)$ with

$$J(u, \Gamma) = \frac{1}{2} \int_D (u - u_d)^2 d\mathbf{x} + \frac{\mu}{2} \int_{D \setminus \Gamma} |\nabla u|^2 d\mathbf{x} + \nu \int_\Gamma 1 d\mathcal{H}_1.$$





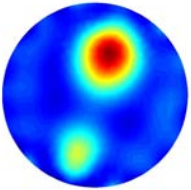
Possible simplification: piecewise constant Mumford-Shah model:

$$u = \sum_{i=1}^{m} u_i \chi_{\Omega_i}, \quad \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j; \quad \Omega_i \subset D \ \forall i.$$

Shape optimization.

- $f(\mathcal{D}) = J(u(\mathcal{D}), q(\mathcal{D}), \mathcal{D})$ where $u(\mathcal{D})$ solves PDE depending on $\mathcal{D}$.

  Electrical impedance tomography.



courtesy: Dept. Physics, Univ. Kuopio

  - Given: Surface measurements associated with electric currents f on Σ.

  - Find: Electrical properties (conductivity) in the interior of the body.

Shape optimization.

Given scenarios $f_i(\mathbf{x}) \in H^{-1/2}(\Omega)$ and associated measurements $m_i \in L^2(\Sigma)$, $i = 1, \ldots, M$, consider

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\sum_{i=1}^{M} \|u_i - m_i\|^2_{L^2(\Sigma)} + \alpha R(q) \\
\text{subject to } & \nabla \cdot (q \nabla u_i) = 0 \text{ in } H^1(\Omega)', \\
& q\partial_n u_i = f_i \text{ on } \Sigma, \\
& \int_\Sigma u_i \, ds = 0, \quad i = 1, \ldots, M.
\end{aligned}$$

- Assumption: q is piecewise constant: $q(\mathbf{x}) = \sum_{j=1}^{m} q_j \chi_{\Omega_j}(\mathbf{x})$.
- Appropriate regularization: Total variation (TV)

$$R(q) = \int_\Omega |\nabla q| = \sum_{j=1}^{m} q_j \int_{\Omega_j} |\nabla \chi_{\Omega_j}| = \sum_{j=1}^{m} q_j \mathrm{Per}(\Omega_j).$$

Issues in solver design.

- Shape and topological sensitivity: How to differentiate a function with respect to (sufficiently regular) sets? Changes in shape vs. changes in topology.
- How to represent geometry in a practical way (avoiding parameterizations etc.)?
- Are there analogues of, e.g., the steepest descent, gradient-related or Newton-type methods for minimizing a shape functional?
- How to "transport" geometry from iteration to iteration?
- Numerical realization?

Shape sensitivity

[Murat, Simon], [Delfour, Sokolowski, Zolesio]. Let V be some sufficiently regular vector field. Define $T_t(V)(\mathbf{x}) = \mathbf{x}(t)$ as the solution of

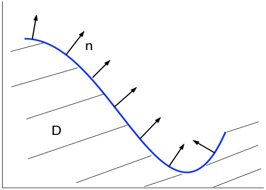
$$\frac{d\mathbf{x}}{dt}(t) = V(t, \mathbf{x}(t)), \quad 0 < t < \tau, \quad \mathbf{x}(0) = \mathbf{x}.$$

Then $\mathcal{D}_t = \mathcal{D}_t(V) := T_t(V)(\mathcal{D})$, $T_0(V)(\mathcal{D}) = \mathcal{D}$.

- Eulerian semi-derivative df of f at $\mathcal{D}$.

$$|f(\mathcal{D}_t) - f(\mathcal{D}) - t \cdot df(\mathcal{D}; V)| = o(t)$$

for $t \downarrow 0$.



- f shape diff. at $\mathcal{D}$: $df(\mathcal{D}; V) = \langle G(\mathcal{D}), V(0) \rangle$.
- In case $\mathcal{D} = \Omega$, there exists $g(\Gamma)$ such that

$$G(\Omega) = \gamma^\star (\mathbf{g}(\Gamma) \cdot \mathbf{n}) .$$

(Hadamard-Zolesio structure theorem)

Shape sensitivity

- Second Eulerian semi-derivative in direction (V, W):

$$d^2 f(\mathcal{D}; V; W) = \lim_{t \downarrow 0} \frac{df(\mathcal{D}_t(W); V) - df(\mathcal{D}; V)}{t}$$

- f twice shape differentiable at $\mathcal{D}$, if $d^2 f(\mathcal{D}; V; W)$ exists for all admissible V and W, and $(V; W) \mapsto d^2 f(\mathcal{D}; V; W)$ is bilinear and continuous. Bilinear form denoted by h.
- Let $H(\mathcal{D}) \in (D(\mathbb{R}^m, \mathbb{R}^m) \otimes D(\mathbb{R}^m, \mathbb{R}^m))'$ denote the vector distribution associated with h, *i.e.*,

$$d^2 J(\mathcal{D}; V; W) = \langle H(\mathcal{D}), V \otimes W \rangle = h(V, W),$$

where $(V \otimes W)_{ij}(x, y) = V_i(x) W_j(y)$ for all $1 \leq i, j, \leq m$. Then $H(\mathcal{D})$ is called the shape Hessian of f at $\mathcal{D}$.

Shape sensitivity

Let $\mathcal{D} = \Omega$ and $\Gamma = \partial\Omega$.

- H has its support in $\Gamma \times \Gamma$, i.e., $\exists$ a vector distribution $h_{\Gamma \oplus \Gamma}(\Omega)$ such that for all V, W there holds

$$\langle h_{\Gamma \oplus \Gamma}(\Omega), (\gamma_\Gamma(V)) \oplus (\gamma_\Gamma(W) \cdot \mathbf{n}) \rangle_\Gamma = d^2 f(\Omega; V; W).$$

  Here $(\gamma_\Gamma(V)) \oplus (\gamma_\Gamma(W) \cdot n)$ is defined as the tensor product

$$((\gamma_\Gamma(V)) \oplus (\gamma_\Gamma(W) \cdot \mathbf{n}))_i\,(x, y) = (\gamma_\Gamma(V_i))(x)\,((\gamma_\Gamma W) \cdot \mathbf{n})\,(y)$$

  for $x, y \in \Gamma$.

Gradient related directions

- Let $B(\mathcal{D};\cdot,\cdot)$ be positive-definite bilinear form. Then the solution $D(\mathcal{D})$ of

$$B(\mathcal{D};V,D(\mathcal{D})) = -df(\mathcal{D};V) \quad \forall V$$

satisfies

$$\langle G(\mathcal{D}),D(\mathcal{D})\rangle < 0$$

if $df(\Omega;V)\neq 0$; $D(\mathcal{D})$ is called shape gradient related.

- Application: Preconditioning of the shape gradient flow by the Laplace-Beltrami operator

$$\int_\Gamma \nabla_\Gamma V_{|\Gamma}\cdot\nabla_\Gamma d(\Gamma)dS = -\int_\Gamma g(\Gamma)\cdot V_n dS,$$

with $V_n := (V\cdot\mathbf{n})|_\Gamma$ for all V; $D(\Omega)=\gamma^\star(d(\Gamma)\cdot\mathbf{n})$.

Line search.

- Given $\mathcal{D}$ compute $\mathcal{D}_t = T_t(D(\mathcal{D}))(\mathcal{D})$ such that

$$f(\mathcal{D}_t) - f(\mathcal{D}) \leq \nu t \langle G(\Omega), D(\Omega) \rangle < 0 \qquad \text{(A)}$$

  with $0 < \nu < 1$.
- t-search by bisection or, more advanced, interpolation schemes.

Theorem: *For given $\mathcal{D}$ there always exists $t_{\mathcal{D}} > 0$ such that* (A) *is satisfied, i.e., the t-search is a finite process.*

Line search; cont.

Stationarity.

- Assume that $t^k := t_{\mathcal{D}^k} \geq \underline{t} > 0$ for all $k \in \mathbb{N}$, and
- $D^k := D(\mathcal{D}^k)$ is uniformly shape gradient related

$$\langle G^k, D^k \rangle \leq -\delta \| G^k \|^2 \quad \forall k$$

  with $\delta > 0$, $G^k := G(\mathcal{D}^k)$, and
- $\{f^k\}$, $f^k := f(\mathcal{D}^k)$, is bounded from below.

Then $f^{k+1} - f^k \leq -\delta \| G^k \|^2 < 0$ for all k, and, thus,

$$|f^{k+1} - f^k| \downarrow 0 \quad \text{and} \quad \| G^k \| \to 0.$$

Variable metric direction.

- Second Eulerian derivative.

$$|df(\mathcal{D}_t(W); V) - df(\mathcal{D}; V) - t \cdot d^2 f(\mathcal{D}; V, W)| = o(t).$$

  for $t \downarrow 0$ and for all velocity fields V, W.
- Newton equation:

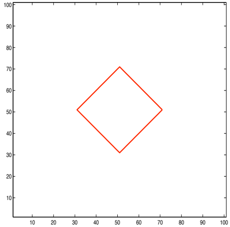
$$d^2 f(\mathcal{D}^k; V, D^k) = -df(\mathcal{D}^k; V) \quad \forall V.$$

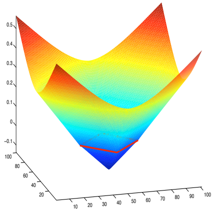
- More general: $B(\mathcal{D}^k; V, D^k)$ uniformly positive-def.
- Preconditioner for the negative shape gradient varies in every iteration:

Variable metric method.

Level set method

- Choose $\Gamma = \partial\mathcal{D}$ to be the zero level set of a Lipschitz fctn $\phi : \mathbb{R}^n \times \mathbb{R}_0^+ \to \mathbb{R}$ (e.g., signed distance fctn)





- Requirement: $\phi(\mathbf{x}(t), t) = 0 \quad \forall \mathbf{x}(t) \in \mathcal{D}_t$. Yields

$$\phi_t(\mathbf{x}(t), t) + \nabla_x \phi(\mathbf{x}(t), t) \cdot \dot{\mathbf{x}}(t) = 0.$$

  Note: $\mathbf{n} = \nabla\phi / |\nabla\phi|$. Choose $\dot{\mathbf{x}}(t) = F \cdot \mathbf{n}$, F a scalar fctn.

- Particular choice of T_t by using level sets:

$$\phi_t + F|\nabla\phi| = 0, \quad \phi(0) = \phi^k, \{\phi^k = 0\} = \partial\mathcal{D}^k.$$

Level set method

Extension.

- Choose $F|_{\Gamma^k} = d(\Gamma^k)$.
- Extension velocity:

$$\langle \nabla d_{\text{ext}}^k, \nabla \phi^k \rangle = 0 \text{ on } \mathbb{R}^2, \quad d_{\text{ext}}^k|_{\Gamma^k} = d(\Gamma^k),$$

  with ϕ^k the signed distance to Γ^k.
- Note: $V_{d_{\text{ext}}^k} := d_{\text{ext}}^k \nabla \phi^k$ satisfies

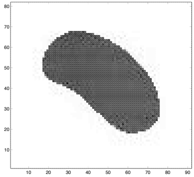
$$\langle V_{d_{\text{ext}}^k}, \mathbf{n} \rangle = d(\Gamma^k).$$

Applications.

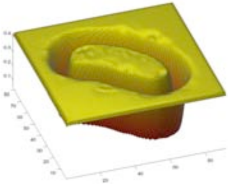
For edge detector based image segmentation one seeks to locally minimize the functional

$$f(\Gamma) = \int_\Gamma g_I \, dS + \nu \int_\Omega g_I \, d\mathbf{x}.$$

Here g_I is an edge detector for the edges in the original image u_d.



Image!



Edge detector!

Applications

Utilizing standard shape sensitivity analysis we obtain

$$df(\Gamma; V) = \langle G(\Gamma), V\rangle = \int_\Gamma \left\langle \left(\frac{\partial g_I}{\partial n} + g_I\,(\kappa+\nu)\right)\mathbf{n}, V \right\rangle dS.$$

The Newton-type speed function $d(\Gamma)$ solves

$$\boxed{\begin{aligned}\int_\Gamma \Big[\Big(\frac{\partial^2 g_I}{\partial n^2} + (2\kappa+\nu)\,\frac{\partial g_I}{\partial n} + \nu\,\kappa\,g_I\Big)\,d(\Gamma)\,V_n + g_I\langle\nabla_\Gamma d(\Gamma), \nabla_\Gamma V_n\rangle\Big] \\ = -\int_\Gamma \Big(\frac{\partial g_I}{\partial n} + (\kappa+\nu)\,g_I\Big)\,V_n\end{aligned}}$$

with $V_n = (V\cdot\mathbf{n})_{|\Gamma}$.

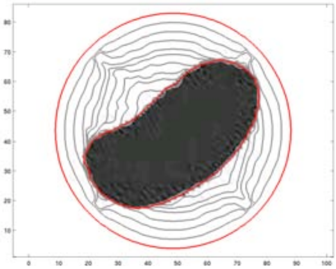
Applications: Algorithm.

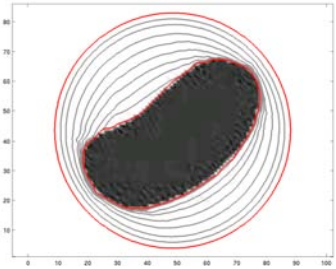
Shape Newton-Algorithm with narrow band.

1. **Initialization.** Choose Γ_0. Compute the initial signed distance function ϕ^0 such that Γ_0 is its zero level set; set $k = 0$. Choose a bandwidth $w \in \mathbb{N}$, $\nu \in \mathbb{R}$.
2. **Newton direction.** Find zero level set Γ_k of ϕ^k. Solve the modified Newton system $\longrightarrow$ Newton-type direction d^k.
3. **Extension.** Extend d^k to band around Γ_k with bandwidth w $\longrightarrow d^k_{ext}$.
4. **Update.** Perform a time step in the level set equation with speed function d^k_{ext} to update ϕ^k on the band $\longrightarrow \hat{\phi}^{k+1}$.
5. **Reinitialization.** Reinitialize $\hat{\phi}^{k+1}$ such that ϕ^{k+1} is signed distance function with zero level set given by the one of $\hat{\phi}^{k+1}$. Set $k = k + 1$ and go to (2).

Numerical Results.

Example 1





Steepest descent!

Newton-type direction!

Numerical Results.

Iteration history for Newton-type direction.

k	Δt^k	Δt^k_{CFL}	f^k_h	$f^k_{h,r}$
1	0.00027	0.00014	67.71894	67.73983
2	0.00916	0.00458	63.62859	63.58714
3	0.05119	0.01462	55.69355	55.30486
4	0.07655	0.02187	45.59301	45.34222
5	0.11608	0.03317	37.06772	36.81020
6	0.16018	0.04577	28.19008	27.54977
7	0.20494	0.05856	16.41064	15.95286
8	0.31020	0.08862	9.73240	9.92598
9	0.34469	0.09848	4.01012	3.83231

Numerical Results

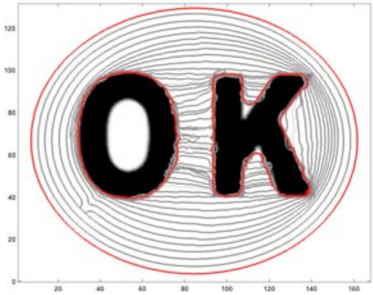
Comparison of Algorithms.

	Newton $\nu = 0$ LS	gradient $\nu = 1$ LS	gradient $\nu = 1$ no LS	gradient $\nu = 0$ LS
# it.	9	13	31	327

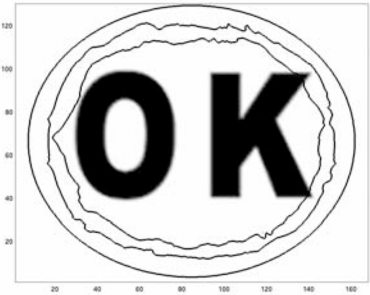
LS ... line search

Numerical Results; cont.

Example 2.



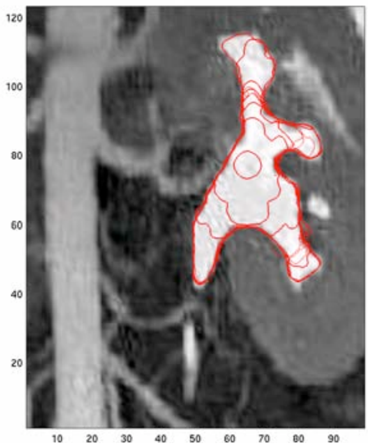
Newton-type direction!



Steepest descent!

Numerical Results; cont.

Example 3.



Newton-type direction!

Narrow band technique; Signed distance fctn on narrow band.

Active contours without edges.

Given: Gray value image $u_d : D \to \mathbb{R}$ (noisy and/or blurred) with $D = (0,1) \times (0,1)$.

Aim: Find denoised and deblurred approximation u to the given data u_d and a set $\Gamma \subset D$ – the *edge set* of the given image u_d – as the minimizer of the Mumford-Shah functional

$$J(u,\Gamma) = \int_D |u - u_d|^2 \, d\mathbf{x} + \frac{\mu}{2} \int_{D \setminus \Gamma} |\nabla u|^2 \, d\mathbf{x} + \nu \int_\Gamma 1 \, d\mathcal{H}_1.$$

Here $\mu, \nu \geq 0$, and $\mathcal{H}_1$ denotes the 1-dimensional Hausdorff measure.

Active contours

- We consider

$$\Gamma = \partial\Omega_1 = \{\mathbf{x} \in D : \phi(\mathbf{x}) = 0\}, \quad \Omega_1 = \{\mathbf{x} \in D : \phi(\mathbf{x}) < 0\}$$

with $\Omega_1 \subset D$ open.

$$\Omega_2 = D \setminus \overline{\Omega_1} = \{\mathbf{x} \in D : \phi(\mathbf{x}) > 0\}$$

- Under suitable assumptions we have

$$\boxed{\inf_{(u,\Gamma)\in H^1(D\setminus\Gamma)\times\mathcal{E}} J(u,\Gamma) = \inf_{\Gamma\in\mathcal{E}} \min_{u\in H^1(D\setminus\Gamma)} J(u,\Gamma).}$$

$\mathcal{E}$ denotes the set of admissible edges.

Active contours

- Set $u_k = u|_{\Omega_k}$ for $k = 1, 2$.
- Note: $u \in H^1(D \setminus \Gamma) \Leftrightarrow u_k \in H^1(\Omega_k)$ for $k = 1, 2$.
- The solution $u(\Gamma) = u_1(\Gamma)\chi_{\Omega_1} + u_2(\Gamma)\chi_{\Omega_2}$ to the inner minimization is then given as the solution to the optimality system

$$\int_{\Omega_k} \big(u_k(\Gamma)\,\varphi + \mu\langle\nabla u_k(\Gamma), \nabla\varphi\rangle\big)\, d\mathbf{x} = \int_{\Omega_k} u_d\,\varphi\, d\mathbf{x}$$

for all $\varphi \in H^1(\Omega_k)$ and for $k = 1, 2$.

Active contours.

- Weak form of the Neumann problem for the Helmholtz equation

$$\left\{ \begin{array}{l} -\mu\Delta u_k(\Gamma) + u(\Gamma) = u_d \text{ on } \Omega_k \\ \frac{\partial u_k(\Gamma)}{\partial n_k} = 0 \text{ on } \partial\Omega_k \end{array} \right.$$

for $k = 1, 2$.

Active contours.

- Remaining shape optimization problem.

minimize

$$f(\Gamma) = \sum_{k=1}^{2} \int_{\Omega_k} \left(\frac{1}{2} |u_k(\Gamma) - u_d|^2 + \frac{\mu}{2} |\nabla u_k(\Gamma)|^2 \right) \, d\mathbf{x} + \nu \int_{\Gamma} 1 \, d\mathcal{H}_1$$

over $\Gamma \in \mathcal{E}$.

Active contours.

- Let $V_F = F\nabla\phi_\Gamma$ with a scalar function F.

  Eulerian derivative of f:

  $$df(\Gamma; V_F) = \int_\Gamma \left(\frac{1}{2}[\![|u-u_d|^2]\!] + \frac{\mu}{2}[\![|\nabla_\Gamma u(\Gamma)|^2]\!] + \nu\kappa\right) F \, d\mathcal{H}_1,$$

  where

  $$[\![|u-u_d|^2]\!] = |u_1 - u_d|^2 - |u_2 - u_d|^2,$$
  $$[\![|\nabla u(\Gamma)|^2]\!] = |\nabla u_1(\Gamma)|^2 - |\nabla u_2(\Gamma)|^2$$

  denote the jumps of $|u-u_d|^2$ and $|\nabla u|^2$, respectively, across Γ.

Shape Hessian.

- $d^2 f(\Gamma; F; G) =$

$$\int_\Gamma \Big[\frac{1}{2}\Big(\kappa\left([\![|u-u_d|^2]\!] - \mu[\![|\nabla_\Gamma u|^2]\!]\right) + \frac{\partial}{\partial n}[\![|u-u_d|^2]\!]\Big)\, G$$
$$+ [\![(u-u_d)\, u'_G]\!] + \mu[\![\langle \nabla u, \nabla u'_G\rangle]\!] - \nu \Delta_\Gamma G\Big]\, F\, d\mathcal{H}_1.$$

- The shape derivative u'_G solves Helmholtz problem

$$\begin{cases} -\mu \Delta u'_{k,G} + u'_{k,G} = 0 \text{ on } \Omega_k \\ \dfrac{\partial u'_{k,G}}{\partial n_1} = \operatorname{div}_\Gamma(G\, \nabla_\Gamma u_k) + \dfrac{1}{\mu}(u_d - u_k)\, G \text{ on } \Gamma, \end{cases}$$

- Shape Hessian evaluation too expensive!!!

Descent direction and PCG.

- Let $B(\Gamma_k; V_F; V_G)$ denote the shape Hessian or a uniformly positive-definite approximation. A descent direction d^k for f at Γ_k is obtained as solution to

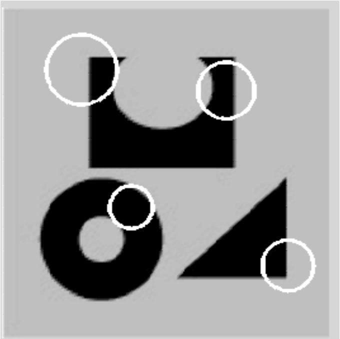
$$B(\Gamma_k; V_F; d^k) = -df(\Gamma_k; V_F) \quad \forall F$$

by means of the preconditioned conjugate gradient method, i.e., d^k satisfies

$$\langle d^k, g^k \rangle_\Gamma < -\delta \| g^k \|_\Gamma^2.$$

- $\Rightarrow$ allows to replace constant time-stepping (CFL-condition) with a line search technique.

Numerical results.



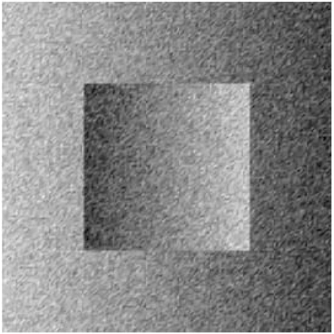
Initialisation!

Segmented image

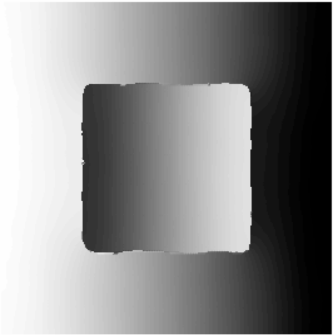
15 Iterations!

Denoising.

Denoising and simultaneous segmentation.



Data!

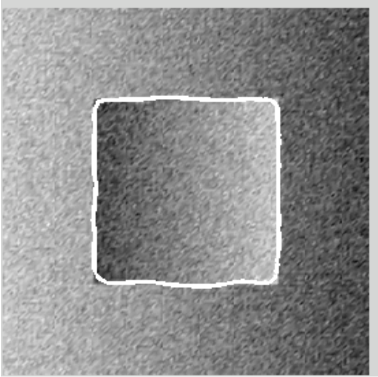


Denoised image

5 Iterations!

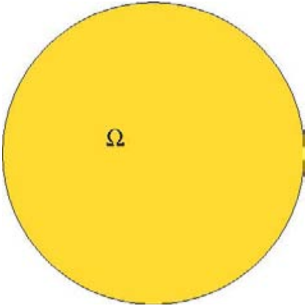
Denoising.

Segmentation result.



Topological derivative

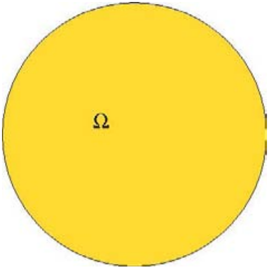
- Often number of connected components (topology) unknown.
- Study sensitivity with respect to topological changes.



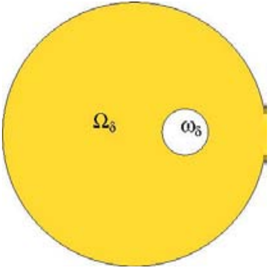
$$J(\Omega) = \int_{\Omega} (u_{\Omega}(\mathbf{x}) - u_d(\mathbf{x}))^2 \, d\mathbf{x}$$

with u_{Ω} the solution of a PDE on Ω.

Topological derivative



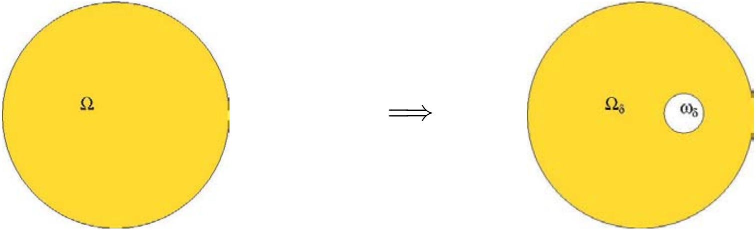
$\Longrightarrow$



$$J(\Omega) = \int_{\Omega} (u_{\Omega}(\mathbf{x}) - u_d(\mathbf{x}))^2 \, dx$$

$$J(\Omega_\delta) = \int_{\Omega_\delta} (u_{\Omega_\delta}(\mathbf{x}) - u_d(\mathbf{x}))^2 \, d\mathbf{x}$$

Topological derivative



Topological derivative ([Eschenauer, Schumacher], [Sokolowski, Zochowski], [Masmoudi et al.]). Let ω_δ be a ball of radius δ and center $\mathbf{x}_0 \in \Omega$, $\delta > 0$. $\Omega_\delta = \Omega \setminus \omega_\delta$. When, for $\delta \to 0$,

$$J(\Omega_\delta) = J(\Omega) + \rho(\delta)\mathcal{T}(\mathbf{x}_0) + o(\rho(\delta)),$$

$\mathcal{T}(\mathbf{x}_0)$ is the topological derivative at $\mathbf{x}_0$ and $\rho(\delta) \to 0$, $\rho(\delta) > 0$.

How to compute the topological derivative?

- Examples of shape functionals

$$J_1(\Omega) = |\Omega|, \qquad J_2(\Omega) = |\partial\Omega|$$

$$J_3(\Omega) = \int_\Omega (u_\Omega(\mathbf{x}) - u_d(\mathbf{x}))^2 \, dx, \qquad J_4(\Omega) = \int_\Omega |\nabla u_\Omega(\mathbf{x})|^2 \, d\mathbf{x}$$

where u_Ω is the solution of a PDE defined in the domain Ω.

- Examples of topological derivatives:

$$J_1(\Omega \setminus B(\mathbf{x}_0, \delta)) = |\Omega \setminus B(\mathbf{x}_0, \delta)| = J_1(\Omega) - \pi\delta^2,$$
$$J_2(\Omega \setminus B(\mathbf{x}_0, \delta)) = |\partial\Omega \cup \partial B(\mathbf{x}_0, \delta)| = J_2(\Omega) + 2\pi\delta$$

Thus, $\mathcal{T}_1(\mathbf{x}_0) = -1$, $\rho_1(\delta) = \pi\delta^2$ and $\mathcal{T}_2(\mathbf{x}_0) = 1$, $\rho_2(\delta) = 2\pi\delta$.

Topological derivative: example

$$J_3(\Omega) = \int_\Omega (u_\Omega(\mathbf{x}) - u_d(\mathbf{x}))^2 \, d\mathbf{x}, \qquad J_3(\Omega_\delta) = \int_{\Omega_\delta} (u_{\Omega_\delta}(\mathbf{x}) - u_d(\mathbf{x}))^2 \, d\mathbf{x}$$

$$\begin{aligned} -\Delta u_\Omega &= g \text{ in } \Omega, \\ u_\Omega &= 0 \text{ on } \Gamma = \partial\Omega. \end{aligned} \qquad \begin{aligned} -\Delta u_{\Omega_\delta} &= g \text{ in } \Omega_\delta, \\ u_{\Omega_\delta} &= 0 \text{ on } \Gamma, \\ \delta_n u_{\Omega_\delta} &= 0 \text{ on } \partial B(\mathbf{x}_0, \delta). \end{aligned}$$

Topological derivative.

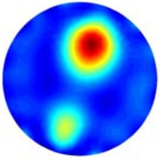
$$J_3(\Omega_\delta) = J_3(\Omega) + \pi\delta^2 \mathcal{T}(\mathbf{x}_0) + o(\pi\delta^2).$$

$$\mathcal{T}(\mathbf{x}_0) = 2\nabla u_\Omega(\mathbf{x}_0) \cdot \nabla p(\mathbf{x}_0) - p(\mathbf{x}_0) g(\mathbf{x}_0) - (u_\Omega - u_d)^2(\mathbf{x}_0).$$

Associated adjoint problem.

$$\begin{aligned} -\Delta p &= 2(u - u_d) \text{ in } \Omega, \\ p &= 0 \text{ on } \Gamma. \end{aligned}$$

EIT.



courtesy: Dept. Physics, Univ. Kuopio

Given $f_i(\mathbf{x})$ and measurements m_i, $i = 1, \ldots, M$, consider

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{M} \|u_i - m_i\|^2_{L^2(\Sigma)} + \alpha \sum_{j=1}^{2} q_j \int_{\Omega_j} |\nabla \chi_{\Omega_j}|$$

$$\text{subject to } \nabla \cdot (q \nabla u_i) = 0 \text{ in } H^1(\Omega)',$$

$$q\partial_n u_i = f_i \text{ on } \Sigma,$$

$$\int_\Sigma u_i ds = 0, \quad i = 1, \ldots, M.$$

Note: $q(\mathbf{x}) = q_1 \chi_{\Omega_1}(\mathbf{x}) + q_2 \chi_{\Omega_2}(\mathbf{x})$; $\Omega_2 = \Omega \setminus \bar{\Omega}_1$.

EIT.

Let $\Omega_2^\delta := \Omega_2 \setminus B(\mathbf{x};\delta)$. Then, for sufficiently small δ, we obtain the expansion:

$$\mathcal{J}(\Omega_2^\delta) = \mathcal{J}(\Omega_2) + \sum_{l=0}^{4} \mathcal{T}_l^{(\delta)}(\mathbf{x}) + \mathcal{O}(\delta^d) + r^{(\delta)}(\mathbf{x}),$$

where $r^{(\delta)}(\mathbf{x})$ denotes a remainder term. The leading term is

$$\mathcal{T}_0^{(\delta)}(\mathbf{x}) = -\delta\beta d^{-1}|S_\delta^{(d-1)}|\nabla w(\mathbf{x}) \cdot \nabla u_2(\mathbf{x}),$$

where $\beta = (q_1 - q_2)/(q_2 + q_1/(d-1))$, and w denotes the adjoint state solving

$$-\Delta w = 0 \text{ in } \Omega, \quad \partial_{n_2} w = (u_2 - m) \text{ on } \Sigma.$$

EIT.

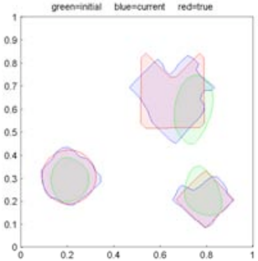
The remaining terms are of higher order in δ, but are important numerically near the boundary Σ:

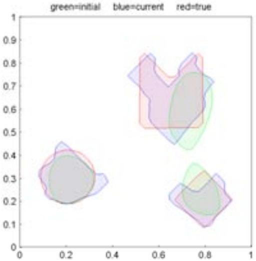
$$
\begin{aligned}
\mathcal{T}_1^{(\delta)}(\mathbf{x}) &= \delta^{2d} \frac{\beta^2}{2(d-1)^2} \sum_{i,j} \partial_i u_2(\mathbf{x})\, \partial_j u_2(\mathbf{x}) \mathcal{I}^{(1)}_{i,j}, \\
\mathcal{T}_2^{(\delta)}(\mathbf{x}) &= \delta^{2(d+2)} \frac{\delta^2}{2d^2} \sum_{i,j,k,l} \partial^2_{ij} u_2(\mathbf{x})\, \partial^2_{kl} u_2(\mathbf{x})\, \mathcal{I}^{(2)}_{i,j,k,l}, \\
\mathcal{T}_3^{(\delta)}(\mathbf{x}) &= \delta^{2d+2} \frac{\beta\delta}{(d-1)d} \sum_{i,j,k} \partial_k u_2(\mathbf{x})\, \partial^2_{ij} u_2(\mathbf{x})\, \mathcal{I}^{(12)}_{i,j,k}, \\
\mathcal{T}_4^{(\delta)}(\mathbf{x}) &= -|\Sigma|^{-1} \left(\sum_k \frac{\delta^d \beta}{d-1} \partial_k u_2(|x) \mathcal{I}^{(\lambda,1)}_k + \sum_{i,j=1} \frac{\delta^{d+2}\delta}{d} \partial^2_{ij} u_2(\mathbf{x})\, \mathcal{I}^{(\lambda,2)}_{i,j} \right)
\end{aligned}
$$

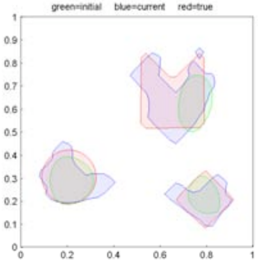
with δ is similar to β. $\mathcal{I}^{\bullet}_{\bullet}$ represent integral terms which can be computed explicitly.

Results.

Reconstructions for 1% (upper left), 3% (upper right) and 5% (lower) noise.







red	...	original;
blue	...	reconstruction;
green	...	initialization.

Finally: Level set relaxation

For $g \in L^p(\Omega)$, with $p > 1$, consider the problem

$$\text{minimize} \quad \int_\Omega g\, u\, d\mathbf{x} + J(u)$$

over $u \in \mathrm{BV}(\Omega; \{0, 1\})$ with the BV-seminorm

$$J(u) = \int_\Omega |\nabla u|.$$

Covers, e.g., binary

- total variation based image denoising (ROF).
- Mumford-Shah based image segmentation (MS).

Exact relaxation

"Convexification of feasible set". Replacing $\mathrm{BV}(\Omega; \{0,1\})$ by $\mathrm{BV}(\Omega; [0,1])$ gives the relaxed problem

$$\text{minimize} \quad \int_\Omega g\, u\, d\mathbf{x} + J(u)$$

over $u \in \mathrm{BV}(\Omega; [0,1])$.

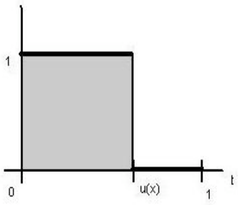
Theorem. *Let $u \in \mathrm{BV}(\Omega; [0,1])$ be a minimizer of the relaxed problem. Then, for almost every $t \in (0,1)$, the function $u^t \in \mathrm{BV}(\Omega; \{0,1\})$ with*

$$u^t(\mathbf{x}) = \begin{cases} 1 & \text{if } u(\mathbf{x}) > t, \\ 0 & \text{else,} \end{cases}$$

is a minimizer of the original problem (with binary constraints).

Exact relaxation

Proof ingredients:

- 

$$\int_0^{u(\mathbf{x})} g(\mathbf{x})\, dt = \int_0^1 g(\mathbf{x})\, u^t(\mathbf{x})\, dt.$$

- Co-area formula. Let $f \in \mathrm{BV}(\Omega)$ and define
$$S_t = \{\mathbf{x} \in \Omega : f(\mathbf{x}) > t\}.$$
Then
$$\int_\Omega |\nabla f| = \int_{-\infty}^{\infty} \int_\Omega |\nabla \chi_{S_t}| dt.$$

Exact relaxation

Proof.

$$
\begin{aligned}
\int_\Omega g\, u\, d\mathbf{x} + J(u) &= \int_\Omega \int_0^{u(\mathbf{x})} g(\mathbf{x}) dt\, d\mathbf{x} + \int_0^1 \int_\Omega |\nabla \chi_{\{u>t\}}| dt \\
&= \int_\Omega \int_0^1 g(\mathbf{x}) u^t(\mathbf{x}) dt\, d\mathbf{x} + \int_0^1 J(u^t) dt \\
&\geq \int_0^1 \left(\int_\Omega g\, v\, d\mathbf{x} + J(v) \right) dt \\
&= \int_\Omega g\, v\, d\mathbf{x} + J(v)
\end{aligned}
$$

where u solves the relaxed and v the "binary" problem. The inequality is due to $u^t \in \mathrm{BV}(\Omega; \{0, 1\})$.

{ROF, MS} and projected gradients

We start by looking at the ROF-model for denoising of blocky images.

$$\text{minimize } \frac{\lambda}{2}\int_\Omega (u-f)^2 d\mathbf{x} + J(u) \text{ over } u \in \mathrm{BV}(\Omega;\{0,1\}).$$

- If $f(\mathbf{x}), u(\mathbf{x}) \in \{0,1\}$ then

$$\frac{\lambda}{2}\int_\Omega (u-f)^2(\mathbf{x})d\mathbf{x} = \int_\Omega u(\mathbf{x})(\frac{\lambda}{2} - \lambda f(\mathbf{x}))d\mathbf{x} + \frac{\lambda}{2}\int_\Omega f(\mathbf{x})^2 d\mathbf{x}.$$

- Setting $g = \frac{\lambda}{2} - \lambda f$, the ROF-minimization is equivalent to our objective.

Movie.

{ROF, MS} and projected gradients

Mumford-Shah-based binary image segmentation.

$$\text{minimize } J_\lambda(u) = \lambda \int_\Omega [u(c_1 - f)^2 + (1-u)(c_2 - f)^2]\, d\mathbf{x} + J(u)$$

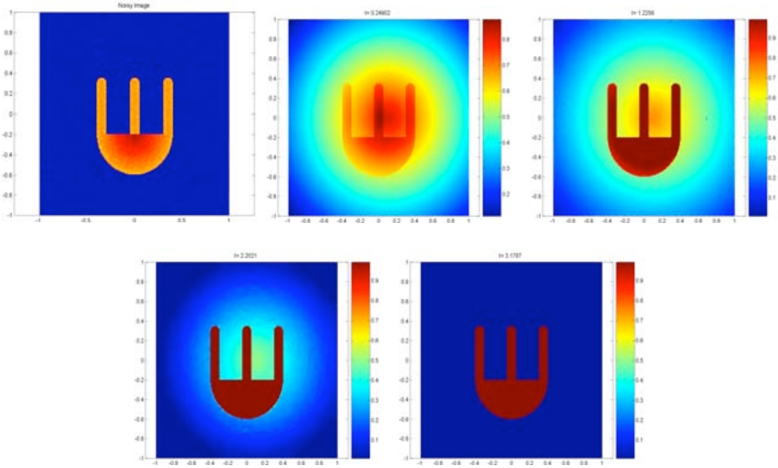
over $u \in \mathrm{BV}(\Omega)(\Omega; \{0,1\})$ and $c_1, c_2 \in \mathbb{R}$.

- For fixed u, $c_1(u)$ and $c_2(u)$ can be computed explicitly.
- Fits into our framework with

$$g = \lambda(c_1 - f)^2 - \lambda(c_2 - f)^2.$$

- Instead of exact minimization w.r.t. u, only a few projected TV-flow steps are performed.
- Numerical example: 256×256, 10% noise, $\lambda = 1000$.

{ROF, MS} and projected gradients



Original, iteration 5, 25, 45, 65.

Movie.