INVERSE PROBLEMS

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Outline

Introductory Example

0 5 10 15 ?

What MECHANISM led to the OBSERVED data?

Interpolation Babylonian Astronomers 300BC

x	0	1	2	3	4
y	0	5	10	15	?

$$p_N(x) = \sum_{j=0}^{N} u_j x^j .$$

$$y(x_i) = p_N(x_i), \quad i = 1, \cdots, 4.$$

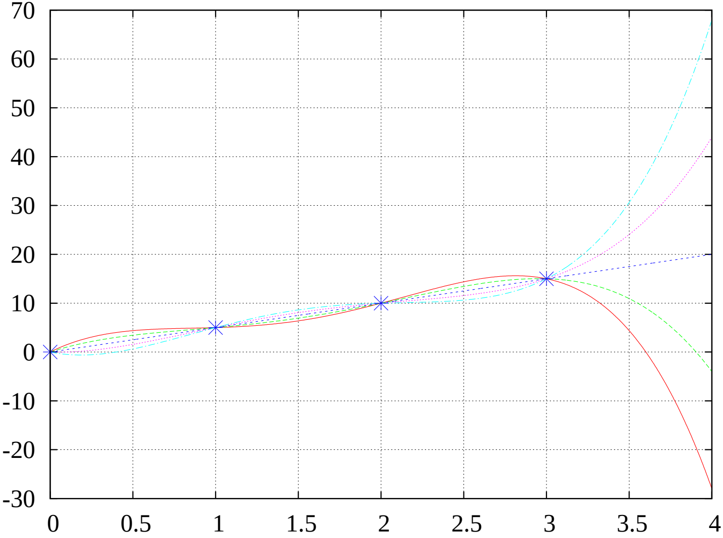
$$N = 3 : \quad y = Au$$

.

$$y, u \in \mathbb{R}^4, \quad A \in \mathbb{R}^{4\times 4}.$$

$$p_3(x) = 5x .$$

Choice of Basis Functions Matters!



Other Complications

What if we change the sequence to:

1 4 9 16 ?

Noisy Data

x	0	1	2	3	4
$y+e$	0	5	10	15	?
y	1	4	9	16	
e	-1	1	1	-1	

Would like to uncover the solution

$$p_3(x) = (1+x)^2 = 1 + 2x + x^2.$$

Neglecting noise in the data leads to a 25% error in prediction.

Least Squares Gauss 1795

$$\operatorname{argmin} \Phi(u), \quad \Phi(u) = \frac{1}{2}\|Au - y\|^2.$$

$$\textit{Normal equations} : A^*Ay = A^*b.$$

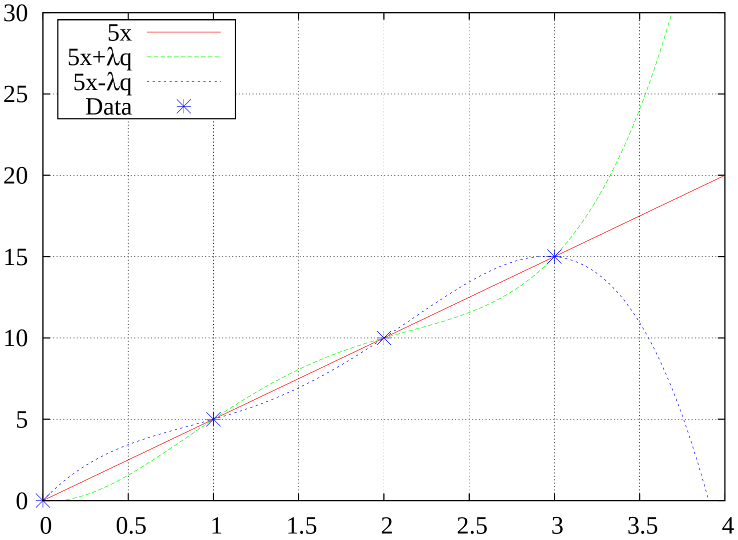
Theorem

For $N \leq 3$ the normal equtions have a unique solution. For $N > 3$ there is an uncountable set of solutions.

$$N = 0 : p_0(x) = 7\frac{1}{2}.$$

$$N = 4 : p_4(x) = 5x + \lambda q(x), \quad \text{any} \quad \lambda \in \mathbb{R}.$$

Least Squares: $N = 4$



Regularized Least Squares

$$\Phi(u) = \frac{1}{2}\|Au - y\|^2.$$

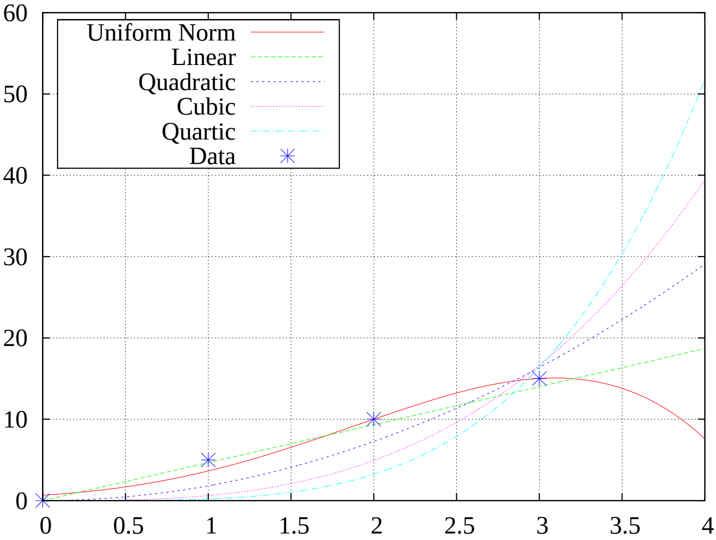
or

$$\operatorname{argmin} I(u) := \Phi(u) + \frac{1}{2}\|\Sigma^{-\frac{1}{2}}u\|^2.$$

Theorem

Assume that Σ is positive definite. Then $I(u)$ has a unique minimum for all dimensions N.

Regularized Least Squares: Effect of Σ



Nonparametric 1970s

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{N} |u(x_i) - y_i|^2.$$

$$I(u) = \Phi(u) + \frac{\lambda}{2}R(u).$$

$$\text{argmin}_{u\in X}\, I(u).$$

Optimization

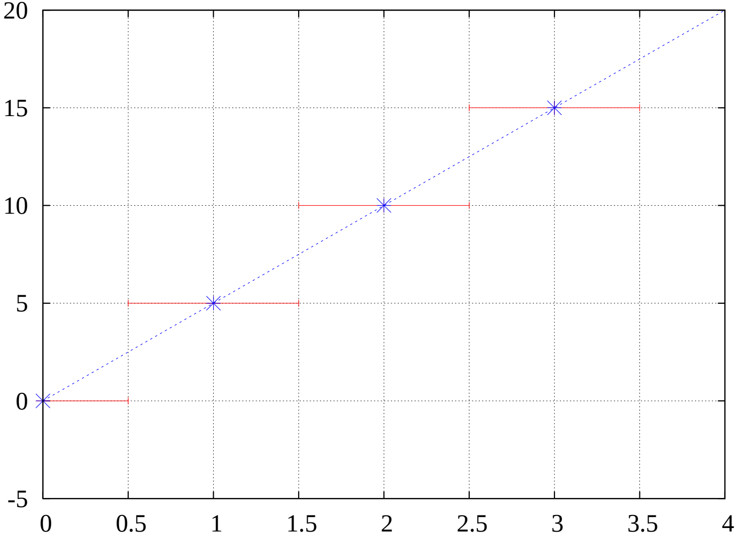
$$\color{red}{Tikhonov\,1963:}\; R(u) = \int \left|\frac{d^s u}{dx^s}\right|^2 dx.$$

$$\color{red}{TotalVariation\,(Mumford/Shah\,1989)/(Osher\,et\,al\,1992):}$$

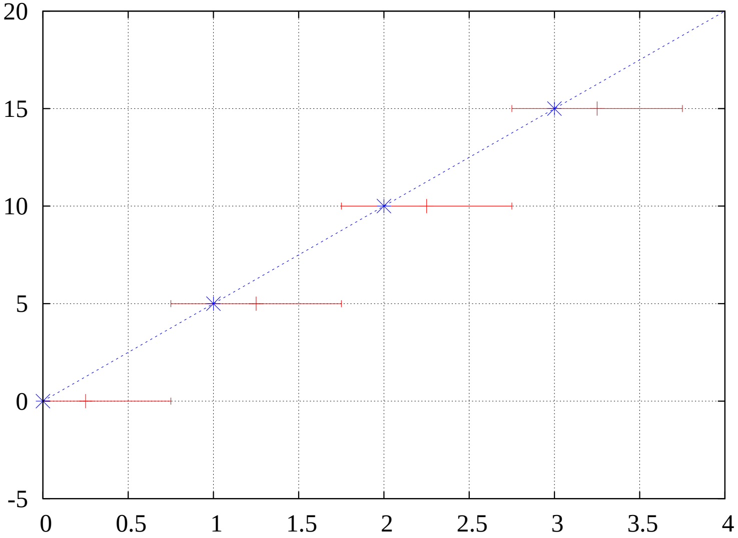
$$R(u) = \int_{x \notin J} \left|\frac{du}{dx}\right| dx + \sum_{x_i \in J} |u(x_i^+) - u(x_i^-)|$$

where J is *the jump set*.

Nonparametric



Nonparametric



Bayes Theorem Bayes 1764

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

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Equating gives Bayes Theorem.

$$\mathbb{P}(A|B) \propto \mathbb{P}(B|A)\mathbb{P}(A).$$

Bayes Theorem in Statistics

$$\mathbb{P}(u|y) \propto \mathbb{P}(y|u)\mathbb{P}(u).$$

Here unknown is u and data is y.

Specifying $\mathbb{P}(y|u)$ requires modelling the error statistics.

Specifying $\mathbb{P}(u)$ requires modelling prior knowledge.

The distribution $\mathbb{P}(u|y)$ summarizes our posterior knowledge.

MOVIE

Bayes and Least Squares

$$\textit{Error model}: y = Au + e, \quad e \sim \mathcal{N}(0, I).$$

$$\textit{Prior}: u \sim \mathcal{N}(0, \Sigma).$$

$$\textit{Posterior}: u \sim \mathcal{N}(m, \mathcal{C}).$$

$$m = \operatorname{argmin}\Big(\frac{1}{2}\|Au - y\|^2 + \frac{1}{2}\|\Sigma^{-\frac{1}{2}}u\|^2\Big).$$

Outline

Signal Processing

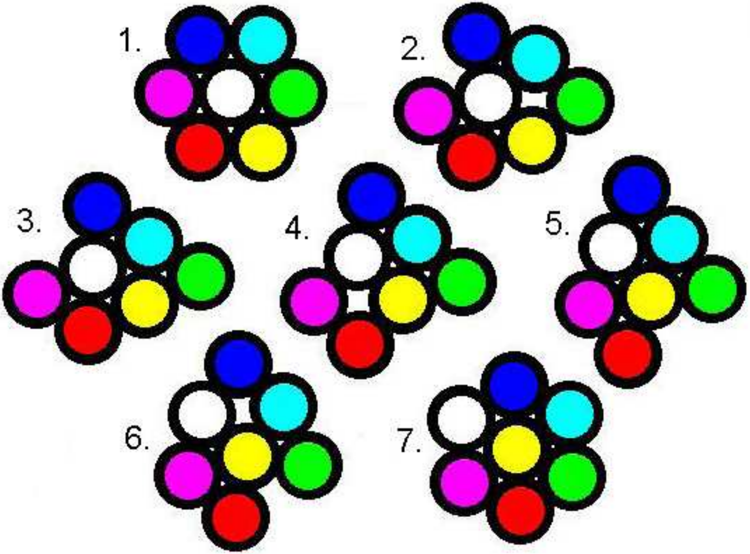
Imagine we wish to find $u(t)$ solving

$$\frac{du}{dt} = u - u^3 + \frac{d}{dt}(\text{NOISE}_1)$$

and we are given $y(t)$ in the form

$$y(t) = \int_0^t u(s)ds + \text{NOISE}_2.$$

Molecular Dynamics



Oceanography

- Consider the Navier-Stokes equation:

$$\frac{dv}{dt} + \nu Av + B(v,v) = f, \quad v(0) = u$$

- Find u (or (u,f)).
- Given noisy Lagrangian observations

$$y_{j,k} = z_j(t_k) + \eta_{j,k}$$
$$\dot{z}_j = v(z_j,t), \quad z_j(0) = z_{j,0}$$

Outline

Random Series

$$S_1 = \sum_{j=1}^{\infty} \frac{1}{j} \quad \textit{diverges}$$

$$S_2 = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \quad ?$$

$$S_3 = \sum_{j=1}^{\infty} \frac{\xi_j}{j}, \quad \xi_j \ i.i.d.\ \mathcal{N}(0,1) \quad ?$$

Random Series

$$S_1 = \sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right)$$
$$= \sum_{k=1}^{\infty}\frac{1}{(2k-1)2k} < \infty.$$

$$\mathbb{E}S_3^2 = \mathbb{E}\sum_{j,k=1}^{\infty}\frac{\xi_j\xi_k}{jk}$$
$$= \sum_{j,k=1}\frac{\delta_{jk}}{jk}$$
$$= \sum_{j=1}^{\infty}\frac{1}{j^2} < \infty.$$

Random Fourier Series

$$\phi_n(x) = \exp\big(2\pi k \cdot x\big).$$

$$\textit{Fourier Series} : u(x) = \sum_{j=1}^{\infty} u_j \phi_n(x).$$

$$\textit{Randomize} : u(x) = \sum_{j=1}^{\infty} \lambda_j \xi_j \phi_n(x), \quad \xi_j \ i.i.d.\ \mathcal{N}(0,1).$$

Function Spaces

What regularity do random Fourier series possess?

$$\textit{Sobolev spaces} : \mathcal{H}^s = \left\{u : \sum_{j=1}^{\infty} j^{2s}|u_j|^2 < \infty\right\}.$$

$$\textit{Holder spaces} : H^s = \{u \in C^{\lfloor s \rfloor} \Big| \sup_{x,y} \frac{|u(x)-u(y)|}{|x-y|^{s-\lfloor s \rfloor}} < \infty\}.$$

Sobolev Regularity

$$u(x) = \sum_{j=1}^{\infty} \lambda_j \xi_j \phi_n(x), \quad \xi_j \ i.i.d. \ \mathcal{N}(0,1).$$

Theorem

$u \in \mathcal{H}^s$ almost surely iff $\sum_{j=1}^{\infty} \lambda_j^2 j^{2s} < \infty$.

The idea:

$$\mathbb{E}\|u\|_{\mathcal{H}^s}^2 = \mathbb{E}\sum_{j=1}^{\infty} \xi_j^2 \lambda_j^2 j^{2s} = \sum_{j=1}^{\infty} \lambda_j^2 j^{2s}.$$

Hölder Regularity

$$u(x) = \sum_{j=1}^{\infty} \lambda_j \xi_j \phi_n(x), \quad \xi_j \ i.i.d. \ \mathcal{N}(0,1).$$

Theorem

(Kolmogorov 1938) Consider random u in dimension d. If there exist $C, \epsilon > 0$ and $\delta > 1$ such that

$$\mathbb{E}|u(x) - u(y)|^{\delta} \leq C|x - y|^{2d+\epsilon}$$

then almost surely $u \in H^s$ for $s < \epsilon/\delta$.

Outline

Background

- Least Squares: "Numerical methods for least squares problems", A. Bjorck, SIAM, 1996.
- Inverse Problems: "Regularization of inverse problems", H.W. Engl, M.Hanke and A. Neubauer, Kluwer, 1996.
- Bayesian Statistics: "Statistical decision theory and Bayesian analysis", J.O. Berger, Springer, 1985.
- Random Functions: "Some random series of functions", J.P. Kahane, CUP, 1994.

My Research

- All papers can be found at:
`http://www.warwick.ac.uk/~masdr/`
- "Inverse Problems: A Bayesian Perspective", A.M. Stuart, Acta Numerica 2010.
- "Bayesian inverse problems for functions and applications to fluid mechanics", S.L. Cotter, M. Dashti, J.C. Robinson, A.M. Stuart, Inverse Problems, 25(2009), 115008.
- Approximation of Bayesian inverse problems for PDEs", S.L. Cotter, M. Dashti and A.M. Stuart, SIAM J. Num. Anal, (48)2010, 322–345.