Shape and topological sensitivities in mathematical image processing

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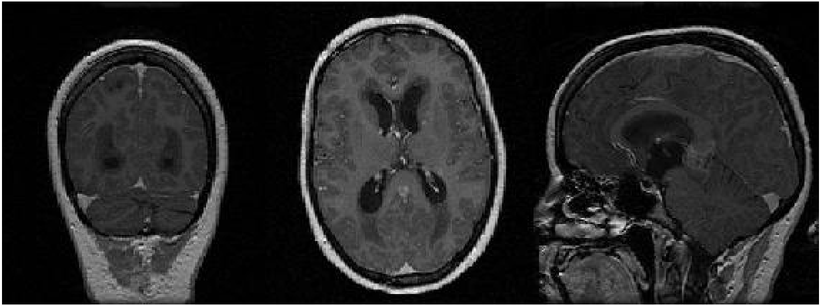
joint work with A. Laurain (U Sao Paolo) and W. Ring (Uni Graz).

Why fast image segmentation?

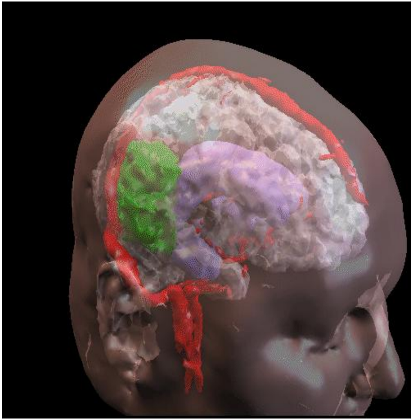
Why fast image segmentation?

Preoperative data - high resolution scans.



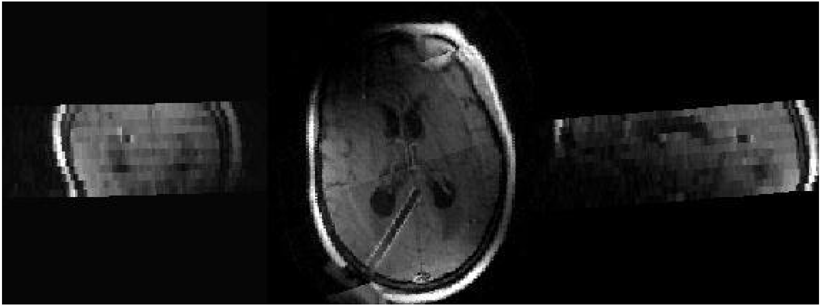
Why fast image segmentation?

Surface rendering of preoperative data.



Why fast image segmentation?

Intraoperative MR scans.



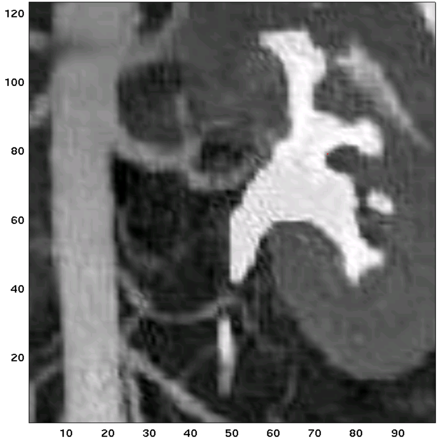
- smaller volume required / available.
- low resolution - compromise: quality vs. acquisition time.

Active contour approaches

- Edge detector based image segmentation.
  - $\rightarrow$ computationally inexpensive;
  - $\rightarrow$ 'edgy' image required.
- Active contours without edges.
  - $\rightarrow$ region bases models - Mumford-Shah;
  - $\rightarrow$ requires PDE solution.

Fundamental task of image segmentation

- Given a (possibly noise corrupted gray scale) image...
- ...find boundary curves of regions with approx. constant gray levels (contours).



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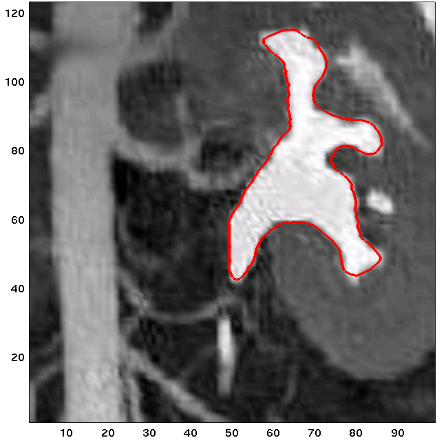


Some existing approaches

- $\Rightarrow$ Global energy principles satisfied by optimal contour.
- $\Rightarrow$ Snakes – active contours as deformable models based on energy minimization along curve. Disadvantage: Depends on parametrization - non geometric (non intrinsic) model.
- $\Rightarrow$ Geodesic active contours combining a geometric model with the energy minimization approach. Parametrization by Euclidean arclength of curve. Curve evolution:

  $$\frac{d\mathcal{C}}{dt} = (g(\mathcal{C})\kappa - \langle \nabla g, \mathcal{N} \rangle)\mathcal{N},$$

  where g is an edge detector.
- $\Rightarrow$ Deformable active contours with contour as *zero-level set* of time dependent function u in terms of geometrically intrinsic formulation. Propagation according to

  $$u_t + F|\nabla u| = 0.$$

Choices of F

- $$F = g\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nu\right)$$

  with ν a ”balloon force”.
- The choice

  $$F = \operatorname{div}\left(g\frac{\nabla u}{|\nabla u|}\right) = g \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \frac{1}{|\nabla u|}\langle \nabla g, \nabla u\rangle,$$

  can be interpreted as the gradient direction of the energy

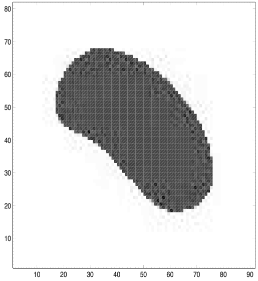
  $$J(\Gamma) = \int_\Gamma g\, dS \qquad (\Gamma \ldots \text{contour}).$$

Edge detector based segmentation

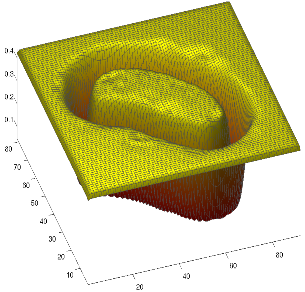
For image segementation one seeks to locally minimize the functional

$$J(\Gamma) = \int_{\Gamma} g_I \, dS + \nu \int_{\Omega} g_I \, dx.$$

Here g_I is an edge detector for the edges in the original image I.



Image!

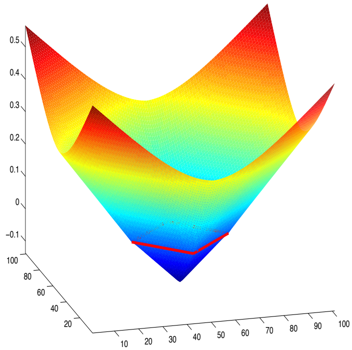


Edge detector!

Signed distance function

The signed distance function b_Ω of a bounded open set $\Omega \subset \mathbf{R}^2$ is defined as

$$b_\Gamma(x) = d_\Omega(x) - d_{\mathbf{R}^2} \setminus \Omega(x).$$



Properties

- $|\nabla b_\Gamma|^2 = 1$ a.e. on $\mathbf{R}^2$ if meas(Γ)=0.
- $\nabla b_\Gamma\big|_\Gamma = n$.
- $\Delta b_\Gamma\big|_\Gamma = \kappa$.
- $b'_\Gamma|_\Gamma = -v_n$ with $v_n = \langle V, n\rangle|_\Gamma$.
- $\Delta b'_\Gamma|_\Gamma = -\Delta_\Gamma v_n$ with $\Delta_\Gamma w = \mathrm{div}_\Gamma(\nabla_\Gamma w)$ the Laplace-Beltrami operator.

Gradient and Newton-type level set flow

$\Rightarrow$ Level set idea and descent flow. Suppose $F : \Gamma \to \mathbf{R}$ is descent direction, e.g. negative shape gradient.

Propagating front formulation for $\Gamma(t)$:

$$\dot{x}(t) = F\big((t), \Gamma(t)\big)\, n(x(t)) \text{ for } x(t) \in \Gamma(t).$$

Equivalent formulation given by level set equation

$$\phi_t + \tilde{F}|\nabla\phi| = 0 \text{ on } \mathbf{R}^2 \times (0, T)$$

where propagating front is zero level set of ϕ, i.e.

$$\Gamma(t) = \{x \in \mathbf{R}^2 \;:\; \phi(x, t) = 0\}.$$

Scalar function $\tilde{F} : \mathbf{R}^2 \times [0, T) \to \mathbf{R}$ chosen such that $\tilde{F}\big|_{\Gamma(t)} = F(\Gamma(t))$.

- Extension velocity. Some freedom in extending $F : \Gamma \to \mathbf{R}$ to $\tilde{F} : \mathbf{R}^2 \times [0, T) \to \mathbf{R}$. In our context preferred: Constructing $\tilde{F}$ as solution to transport equation

$$\langle \nabla \tilde{F}, \nabla b_\Gamma \rangle = 0 \text{ on } \mathbf{R}^2; \quad \tilde{F}\big|_\Gamma = F$$

most appropriate.
Define

$$V_F = \tilde{F} \nabla b_\Gamma .$$

Then $\langle V_F, n \rangle = F$.

- Newton-type speed function. The Newton-type speed function is the solution $F : \Gamma \to \mathbf{R}$ to

$$d^2 J(\Gamma; V_F; V_G) = -dJ(\Gamma; V_G) \text{ for all } G : \Gamma \to \mathbf{R}.$$

1st and 2nd order Eulerian semi-derivs.

$$dJ(\Gamma; V) = \langle D_\Gamma J, V\rangle = \int_\Gamma \left\langle \left(\frac{\partial g_l}{\partial n} + g_l\,(\kappa+\nu)\right)\mathbf{n}, V\right\rangle dS.$$

Newton-type speed function F solves

$$\boxed{\begin{aligned}\int_\Gamma \left[\left(\frac{\partial^2 g_l}{\partial n^2} + (2\kappa+\nu)\,\frac{\partial g_l}{\partial n} + \nu\,\kappa\, g_l\right) F\,G + g_l\langle\nabla_\Gamma F, \nabla_\Gamma G\rangle\right] dS =\\ -\int_\Gamma \left(\frac{\partial g_l}{\partial n} + (\kappa+\nu)\, g_l\right) G\, dS\end{aligned}}$$

$\Rightarrow$ Coercivity (*).

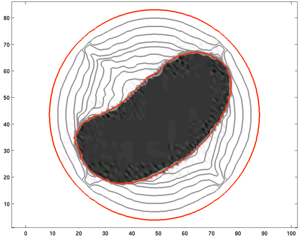
$$\boxed{\begin{aligned}\int_\Gamma \left[\left(\frac{\partial^2 g_l}{\partial n^2} + (2\kappa+\nu)\,\frac{\partial g_l}{\partial n} + \nu\,\kappa\, g_l\right)_+ F\,G + g_l\langle\nabla_\Gamma F, \nabla_\Gamma G\rangle\right] dS =\\ -\int_\Gamma \left(\frac{\partial g_l}{\partial n} + (\kappa+\nu)\, g_l\right) G\, dS\end{aligned}}$$

Shape Newton-Algorithm with narrow band

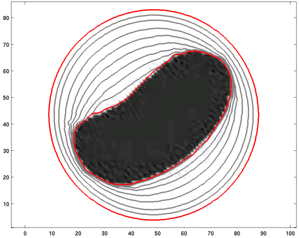
1. **Initialization.** Choose Γ_0. Initialize level set function ϕ^0 such that Γ_0 is zero level set of ϕ^0; set $k = 0$. Choose bandwidth $w \in \mathbf{N}$ and $\nu \in \mathbf{R}$.
2. **Newton direction.** Find zero level set Γ_k of actual level set function ϕ^k. Solve (*) to obtain Newton-type direction F^k.
3. **Extension.** Extend F^k to band around actual zero level set Γ_k with bandwidth w yielding F^k_{ext}.
4. **Update.** Perform time step in level set equation with speed function F^k_{ext} to update ϕ^k on band. Let $\hat{\phi}^{k+1}$ denote this update.
5. **Reinitialization.** Reinitialize $\hat{\phi}^{k+1}$ in order to obtain signed distance function ϕ^{k+1} with zero level set given by zero level set of $\hat{\phi}^{k+1}$. Set $k = k + 1$ and go to (2).

Numerical results.

Example 1



Steepest descent!



Newton-type direction!

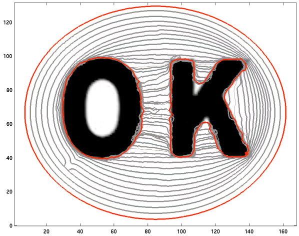
Comparison

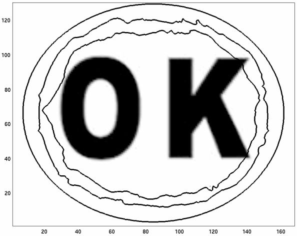
k	Δt^k	Δt^k_{CFL}	J^k_h	$J^k_{h,r}$
1	0.00027	0.00014	67.71894	67.73983
2	0.00916	0.00458	63.62859	63.58714
3	0.05119	0.01462	55.69355	55.30486
4	0.07655	0.02187	45.59301	45.34222
5	0.11608	0.03317	37.06772	36.81020
6	0.16018	0.04577	28.19008	27.54977
7	0.20494	0.05856	16.41064	15.95286
8	0.31020	0.08862	9.73240	9.92598
9	0.34469	0.09848	4.01012	3.83231

Comparison of Algorithms (LS ... line search).

	Newton $\nu = 0$ LS	gradient $\nu = 1$ LS	gradient $\nu = 1$ no LS	gradient $\nu = 0$ LS
# it.	9	13	31	327

Example 2.





Newton-type direction!

Steepest descent!

Active contours without edges

- Given: Gray value image $u_d : D \to \mathbf{R}$ (noisy and/or blurred) with $D = (0,1) \times (0,1)$.
- Aim: Find denoised and deblurred approximation u to given data u_d and a set $\Gamma \subset D$ – the *edge set* of given image u_d – as minimizer of the Mumford-Shah functional

$$J(u,\Gamma) = \int_D |u - u_d|^2 \, dx + \frac{\mu}{2} \int_{D \setminus \Gamma} |\nabla u|^2 \, dx + \nu \int_\Gamma 1 \, d\mathcal{H}_1,$$

with $\mu, \nu \geq 0$, and $\mathcal{H}_1$ the 1-dim. Hausdorff measure.

Consider

$$\Gamma = \partial\Omega_1 = \{x \in D : \phi(x) = 0\}, \quad \Omega_1 = \{x \in D : \phi(\mathbf{x}) < 0\}$$

with $\Omega_1 \subset D$ open.

$$\Omega_2 = D \setminus \overline{\Omega_1} = \{x \in D : \phi(x) > 0\}$$

Under suitable assumptions:

$$\boxed{\inf_{(u,\Gamma)\in H^1(D\setminus\Gamma)\times\mathcal{E}} J(u,\Gamma) = \inf_{\Gamma\in\mathcal{E}} \min_{u\in H^1(D\setminus\Gamma)} J(u,\Gamma).}$$

$\mathcal{E}$ denotes the set of admissible edges.

- Set $u_k = u|_{\Omega_k}$ for $k = 1, 2$.
- Note: $u \in H^1(D \setminus \Gamma) \Leftrightarrow u_k \in H^1(\Omega_k)$ for $k = 1, 2$.
- Solution $u(\Gamma) = u_1(\Gamma)\chi_{\Omega_1} + u_2(\Gamma)\chi_{\Omega_2}$ to inner minimization is given as solution to optimality system

$$\int_{\Omega_k} \left(u_k(\Gamma)\,\varphi + \mu\langle\nabla u_k(\Gamma), \nabla\varphi\rangle\right) dx = \int_{\Omega_k} u_d\,\varphi\,dx$$

  for all $\varphi \in H^1(\Omega_k)$ and for $k = 1, 2$.
- Weak form of Neumann problem for

$$\begin{cases} -\mu\Delta u_k(\Gamma) + u(\Gamma) = u_d \text{ on } \Omega_k \\ \frac{\partial u_k(\Gamma)}{\partial n_k} = 0 \text{ on } \partial\Omega_k \end{cases}$$

  for $k = 1, 2$.

Remaining shape optimization problem.

$$\hat{J}(\Gamma) = \sum_{k=1}^{2} \int_{\Omega_k} \left(\frac{1}{2} |u_k(\Gamma) - u_d|^2 + \frac{\mu}{2} |\nabla u_k(\Gamma)|^2 \right) dx + \nu \int_\Gamma 1 \, d\mathcal{H}_1$$

over $\Gamma \in \mathcal{E}$.

Let $V_F = F \nabla b_\Gamma$ with a scalar function F.

Eulerian derivative of $\hat{J}$:

$$d\hat{J}(\Gamma; V_F) = \int_\Gamma \left(\frac{1}{2} [\![|u - u_d|^2]\!] + \frac{\mu}{2} [\![|\nabla_\Gamma u(\Gamma)|^2]\!] + \nu\kappa \right) F \, d\mathcal{H}_1,$$

where

- $[\![|u - u_d|^2]\!] = |u_1 - u_d|^2 - |u_2 - u_d|^2$;
- $[\![|\nabla u(\Gamma)|^2]\!] = |\nabla u_1(\Gamma)|^2 - |\nabla u_2(\Gamma)|^2$

denote the jumps of $|u - u_d|^2$ and $|\nabla u|^2$, respectively, across Γ.

Shape Hessian

$$d^2\hat{J}(\Gamma; F; G) = \int_\Gamma \Bigg[\frac{1}{2}\Big(\kappa\left([\![|u-u_d|^2]\!] - \mu[\![|\nabla_\Gamma u|^2]\!]\right) + \frac{\partial}{\partial n}[\![|u-u_d|^2]\!]\Big)$$
$$+ [\![(u-u_d)\,u'_G]\!] + \mu[\![\langle\nabla u, \nabla u'_G\rangle]\!] - \nu\Delta_\Gamma G\Bigg]\, F\, d\mathcal{H}_1.$$

The shape derivative u'_G solves

$$\begin{cases} -\mu\Delta u'_{k,G} + u'_{k,G} = 0 \text{ on } \Omega_k \\ \dfrac{\partial u'_{k,G}}{\partial n_1} = \mathrm{div}_\Gamma(G\nabla_\Gamma u_k) + \dfrac{1}{\mu}(u_d - u_k)\, G \text{ on } \Gamma, \end{cases}$$

for $k = 1, 2$.

Shape Hessian evaluation too expensive!!!

Descent direction and PCG.

- Let $B(\Gamma_k; V_F; V_G)$ denote the shape Hessian or a positive definite approximation. A descent direction G_N^k for $\hat{J}$ in Γ_k is obtain as solution to

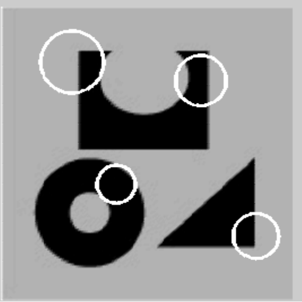
$$B(\Gamma_k; V_F; V_{G_N^k}) = -d\hat{J}(\Gamma_k; V_F) \quad \forall F$$

  by means of the preconditioned conjugate gradient method, i.e., G_N satisfies

$$\langle V_{G_N^k}, V_F \rangle < 0 \quad \forall F.$$

- $\Rightarrow$ allows to replace constant time-stepping (CFL-condition) with a line search technique.

Numerical results.

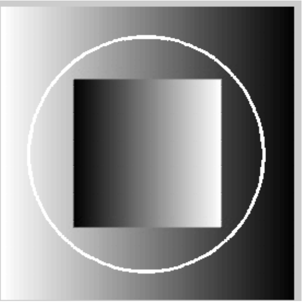


Initialisation!

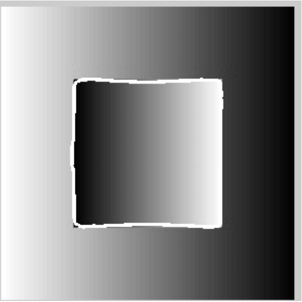
Segmented image

15 Iterations!

Solution of elliptic equation.



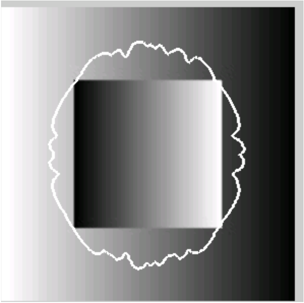
Initialisation!



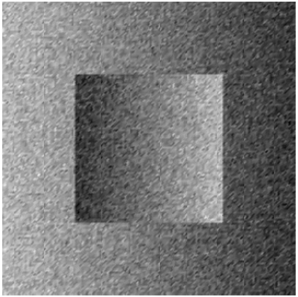
Segmented image

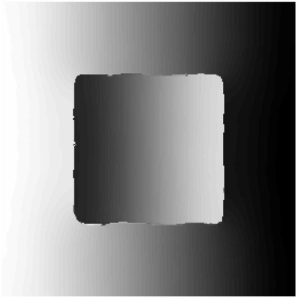
5 Iterations!

Steepest descent method at iteration 8.



Denoising and simultaneous segmentation.



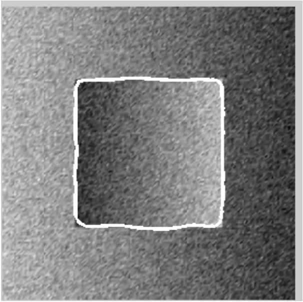


Data!

Denoised image

5 Iterations!

Segmentation result.



Mumford-Shah (MS) model in image segmentation

Given: $f : \Omega \longmapsto \mathbb{R}$ gray-level image, $\Omega \subset \mathbb{R}^2$ bounded;

Find : u, reconstruction of true image, and Γ, the edge set (contours).

$$\min_{u,\Gamma} J(u,\Gamma) = \int_\Omega (f-u)^2 + \mu \int_{\Omega\setminus\Gamma} |\nabla u|^2 + \nu \mathcal{H}^1(\Gamma), \tag{1}$$

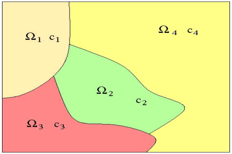
Piecewise constant MS (Chan-Vese)

When u is piecewise constant:

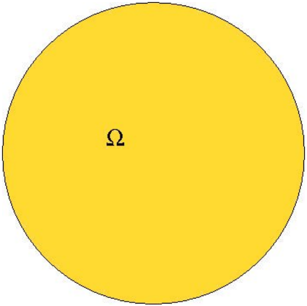
$$\min_{u,\Gamma} J(u,\Gamma) = \int_\Omega (f-u)^2 + \nu \mathcal{H}^1(\Gamma). \tag{2}$$

Algorithm for minimizing (2)

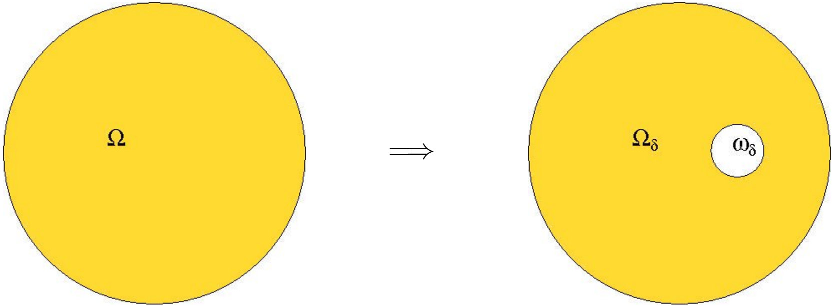
1. Apply topological derivative
2. Apply shape derivative



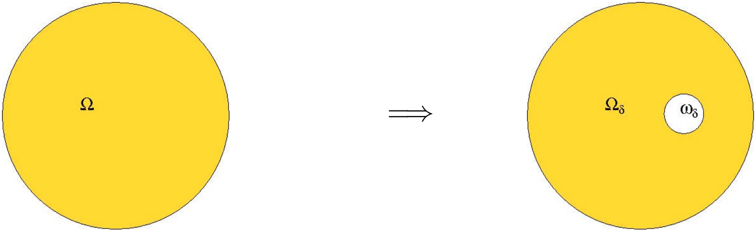
Topological derivative



Topological derivative



Topological derivative



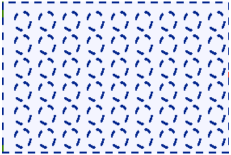
Topological derivative

Let ω_δ be a ball of radius $\delta > 0$ and center $x_0 \in \Omega$. $\Omega_\delta := \Omega \setminus \omega_\delta$.
For $\delta \to 0$ we have

$$J(\Omega_\delta) = J(\Omega) + \rho(\delta)\mathcal{T}(x_0) + o(\rho(\delta)),$$

and $\mathcal{T}(x_0)$ is the topological derivative of J at x_0; $\rho(\delta) \to 0$.

Why use topological derivative?



- Extremely fast in detecting topological structures.
- Independent of initialization.

Why use shape derivative?

- Topological derivative cannot handle perimeter term $\mathcal{H}^1(\Gamma)$.

TOPSHAPE - Algorithm

1. Phase I: Apply TOPological derivative for $\nu = 0$.
2. Phase II: Apply SHAPE derivative for $\nu > 0$.

Agenda

Mumford-Shah functional and topological derivative

- Topological derivative
- Phase I: Algorithm for topological derivative

Mumford-Shah functional and shape derivative + level set framework

- Shape sensitivity analysis
- Phase II: Algorithm for shape derivative

Piecewise constant MS-model (Chan-Vese)

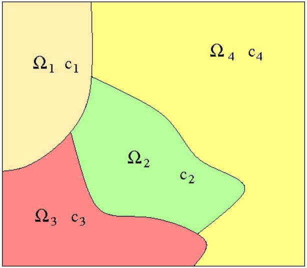


Image f

- $f : \Omega \longmapsto \mathbb{R}$: gray scale image.
- $\Omega \subset \mathbb{R}^2$: corresponding domain $\Omega = \cup_{i=1}^m \Omega_i,\ \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j.$
- $\Gamma = \cup_{i=1}^m \Gamma_i = \cup_{i=1}^m \partial \Omega_i.$
- "colors" $c_i = |\Omega_i|^{-1} \int_{\Omega_i} f(x)\, dx.$
- approximation $u(x) = c_i \ \forall x \in \Omega_i$.

- Note that the $\Gamma_i \cap \Gamma_j$, $j \neq i$, might be non-empty.

For $\nu = 0$ in (2), we obtain

$$J(u, \Gamma) = \int_\Omega (f - u)^2. \tag{3}$$

- Note that $c_i = c_i(\Omega_i)$ for all i.

We can rewrite (3) in terms of Ω_i only:

$$\mathcal{J}_0(\{\Omega_i\}_{i=1}^m) = \sum_{i=1}^m \int_{\Omega_i} (f(x) - c_i)^2 \, dx = J(u, \Gamma). \tag{4}$$

Later we also use

$$\mathcal{J}_\nu(\{\Omega_i\}_{i=1}^m) = \mathcal{J}_0(\{\Omega_i\}_{i=1}^m) + \frac{\nu}{2}\left(\mathcal{H}^1(\partial\Omega) + \sum_{i=1}^m \mathcal{H}^1(\Gamma_i)\right).$$

Shape resp. topology optimization problem

$$
\begin{aligned}
\textbf{Minimize} \quad & \mathcal{J}_\nu(\{\Omega_i\}_{i=1}^m) \\
\textbf{s.t.} \quad & \Omega = \cup_{i=1}^m \overline{\Omega}_i, \\
& \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j, \\
& \Omega_i \text{ measurable } \forall i \in \{1, .., m\}.
\end{aligned} \tag{5}
$$

Theorem

Problem (5) *admits a solution* $\{\Omega_i^*\}_{i \in \{1,..,m\}}$.

Let $\{\Omega_i\}_{i=1}^m$ be given, $\rho > 0$, $x_0 \in \Omega_i$, $B_\rho := B(x_0, \rho)$.

Q: $\mathcal{J}_0$ reduced when moving B_ρ (ρ small) from Ω_i to Ω_j?

Topological derivative

$$\begin{aligned}\mathcal{J}_0(\Omega_1, ..., \Omega_i \setminus B_\rho, ..., \Omega_j \cup B_\rho, ..., \Omega_m) \\ = \mathcal{J}_0(\{\Omega_i\}_{i\in[1,m]}) + \pi\rho^2 \mathcal{T}_{ij}(x_0) + o(\rho^2)\end{aligned}$$

Matrix-valued derivative:

$$\mathcal{T} = \{\mathcal{T}_{ij}\}_{(i,j)\in\{1,...,m\}^2}$$

with $\mathcal{T}_{ii} \equiv 0$ for all $i \in \{1, \ldots, m\}$.

Remove ”material” from Ω_i

Here it is assumed that $\Omega_i \neq \emptyset$

$$c_i(\Omega_i \backslash B_\rho) = c_i(\Omega_i) + \pi|\Omega_i|^{-1}\rho^2\left(c_i(\Omega_i) - \frac{1}{|B_\rho|}\int_{B_\rho} f(x)dx\right) + \mathcal{O}(\rho^2)$$

Add ”material” to Ω_i

If $\Omega_j \neq \emptyset$ then

$$c_j(\Omega_j \cup B_\rho) = c_j(\Omega_j) - \pi|\Omega_j|^{-1}\rho^2\left(c_j(\Omega_j) - \frac{1}{|B_\rho|}\int_{B_\rho} f(x)dx\right) + \mathcal{O}(\rho^2).$$

If $\Omega_j = \emptyset$ then

$$c_j(\Omega_j \cup B_\rho) = \frac{1}{|B_\rho|}\int_{B_\rho} f(x)dx.$$

If $\Omega_j \neq \emptyset$, then we obtain

$$\begin{aligned}
&\mathcal{J}_0(\Omega_1, ..., \Omega_i \setminus B_\rho, ..., \Omega_j \cup B_\rho, ..., \Omega_m) \\
&\quad = \mathcal{J}_0(\{\Omega_i\}_{i\in[1,m]}) + \int_{B_\rho} (f(x) - c_j)^2 - (f(x) - c_i)^2 dx + \\
&\quad 2\pi|\Omega_i|^{-1}\rho^2 \left(c_i(\Omega_i) - \frac{1}{|B_\rho|} \int_{B_\rho} f(x)dx \right) \int_{\Omega_i} (f(x) - c_i)dx - \\
&\quad 2\pi|\Omega_j|^{-1}\rho^2 \left(c_j(\Omega_j) - \frac{1}{|B_\rho|} \int_{B_\rho} f(x)dx \right) \int_{\Omega_j} (f(x) - c_j)dx + \mathcal{O}(\rho^2),
\end{aligned}$$

but since $\int_{\Omega_i}(f(x) - c_i) = \int_{\Omega_j}(f(x) - c_j) = 0$, we obtain:

Topological derivative

$$\mathcal{T}_{ij}(x_0) = (f(x_0) - c_j)^2 - (f(x_0) - c_i)^2.$$

If $\Omega_j = \emptyset$, then we obtain

$$\mathcal{J}_0(\Omega_1, ..., \Omega_i \setminus B_\rho, ..., \Omega_j \cup B_\rho, ..., \Omega_m)$$
$$= \mathcal{J}_0(\{\Omega_i\}_{i\in[1,m]}) + \int_{B_\rho} (f(x) - c_j)^2 - (f(x) - c_i)^2 +$$
$$2\pi|\Omega_i|^{-1}\rho^2 \left(c_i(\Omega_i) - \frac{1}{|B_\rho|} \int_{B_\rho} f(x)dx \right) \int_{\Omega_i} (f(x) - c_i) + o(\rho^2).$$

Since $\int_{\Omega_i}(f(x) - c_i) = 0$ and $c_j(\Omega_j \cup B_\rho) = \frac{1}{|B_\rho|}\int_{B_\rho} f(x)dx$:

Topological derivative

$$\mathcal{T}_{ij}(x_0) = -(f(x_0) - c_i)^2.$$

Observations.

- For $\mathcal{T}_{ij}(x_0) = (f(x_0) - c_j)^2 - (f(x_0) - c_i)^2$, we have

$$\mathcal{T}_{ij}(x_0) = \mathcal{T}_{ik}(x_0) + \mathcal{T}_{kj}(x_0).$$

- For $\mathcal{T}_{ij}(x_0)$ as above, it holds that

$$\mathcal{T}_{ij}(x) < 0 \iff \begin{cases} \text{either} & c_i < c_j \quad \text{and} \quad f(x) > \frac{c_i + c_j}{2}, \\ \text{or} & c_i > c_j \quad \text{and} \quad f(x) < \frac{c_i + c_j}{2}. \end{cases}$$

- Let $d_i := \frac{c_{i-1} + c_i}{2} \quad \forall i \in \{2, .., m\}$ and assume

$$\min(f) \leq c_1^{(0)} < .. < c_i^{(0)} < .. < c_m^{(0)} \leq \max(f),$$

  where $^{(0)}$ refers to initial guess of subsequent algorithm. Let $x \in \Omega_k$ and p be smallest integer such that $\mathcal{T}_{kp}(x) = \min_{l \in \{1,..,m\}} \mathcal{T}_{kl}(x)$. Then

$$d_p < f(x) \leq d_{p+1}. \tag{6}$$

Conversely, if $f(x)$ satisfies (6) for some $\hat{p}$ then

$$\mathcal{T}_{k\hat{p}}(x) = \min_{l \in \{1,..,m\}} \mathcal{T}_{kl}(x).$$

Algorithm

- **Input:** Ω, f, m. **Output**: Ω_i, c_i, $i = 1, .., m$.
- **Initialization:** Choose $\min(f) \leq c_1^{(0)} < .. < c_i^{(0)} < .. < c_m^{(0)} \leq \max(f)$. Set $l = 0$ and $\Omega_i^{(0)} = \emptyset \; \forall i \in \{1, .., m\}$.
- **While**$[(l > 0$ **and** $|\Omega_i^{(l)} \Delta \Omega_i^{(l-1)}| > 0 \; \forall i)$ **or** $l = 0]$

  Compute $d_i^{(l)}$, $i \in \{2, .., m\}$, set $d_1^{(l)} < 0$, $d_{m+1}^{(l)} = \max(f)$.

  Set $\Omega_i^{(l+1)} = \left\{ x \in \Omega \mid d_i^{(l)} < f(x) \leq d_{i+1}^{(l)} \right\} \quad \forall i.$
- **For** $k = 1, .., m$

  if $|\Omega_i^{(l+1)}| > 0$ **then**

  Update $c_k^{(l+1)} = |\Omega_k^{(l+1)}|^{-1} \int_{\Omega_k^{(l+1)}} f(x)\, dx$.

  else

  Choose arbitrary $c_k^{(l+1)}$ outside $[d_{k-1}^{(l)}, d_k^{(l)}]$.

  set $l = l + 1$

Chan-Vese model

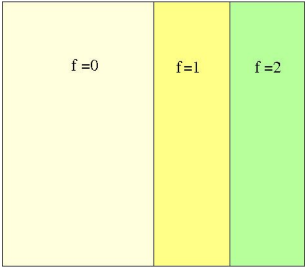


Image f

Piecewise constant Mumford-Shah model

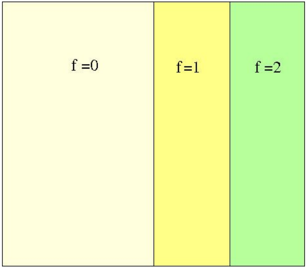
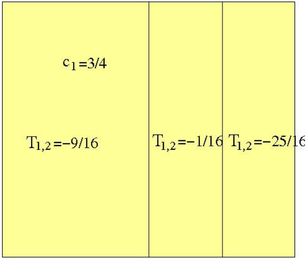
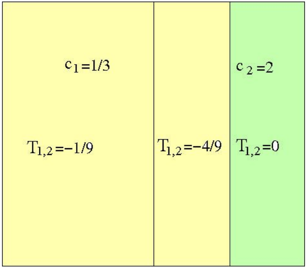


Image *f*



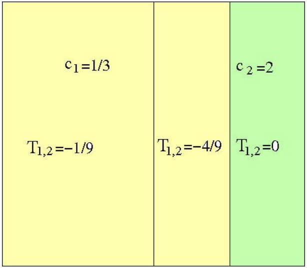
Initialization

Chan-Vese model

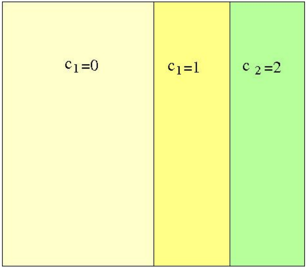


First iteration

Chan-Vese model

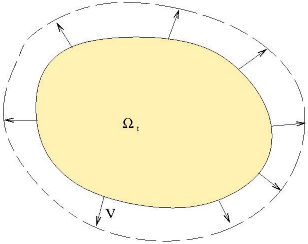


First iteration



Second iteration

Shape derivative



moving domain Ω_t

X: Lagrangian coordinate
$x(t, X)$: Eulerian coordinate

$$\begin{aligned} \frac{d}{dt} x(t, X) &= V(t, x(t, X)) \\ x(0, X) &= X \end{aligned}$$

$T_t(V)(X) = x(t, X)$
$\Omega_t = T_t(V)(\Omega)$: moving domain

$J(\Omega_t)$: shape functional

$dJ(\Omega, V) = \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$

Structure theorem

If Ω is smooth enough there exists ∇J on Γ such that

$$dJ(\Gamma, V) = \langle \nabla J, v_n \rangle_\Gamma,$$

where $v_n(x) = V(0, x) \cdot n(x)$ and $\langle \cdot, \cdot \rangle_\Gamma$ a duality pairing. If duality pairing can be realized as integral over Γ we have

$$dJ(\Gamma, V) = \int_\Gamma \nabla J \, v_n \, d\Gamma, \tag{7}$$

and we are able to use a gradient-descent method by choosing $v_n = -\nabla J$.

Due to the Four-Color Theorem, we choose $m = 4$. Define two level set functions ϕ_1 and ϕ_2 such that

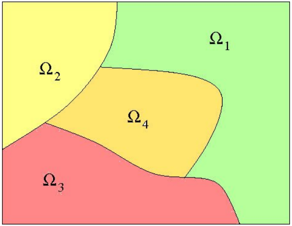
$$\begin{aligned} \Omega_1 \cup \Omega_2 &= \{x \in \Omega \mid \phi_1(x) < 0\} && (8) \\ \Omega_1 \cup \Omega_3 &= \{x \in \Omega \mid \phi_2(x) < 0\} . && (9) \end{aligned}$$

For instance, the set Ω_1 can be deduced by

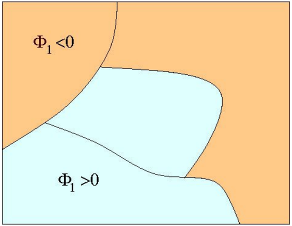
$$\Omega_1 = \{x \in \Omega \mid \phi_1(x) < 0 \text{ and } \phi_2(x) < 0\} .$$

For convenience we also define the sets D_1 and D_2

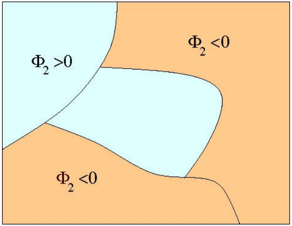
$$\begin{aligned} D_1 &:= \Omega_1 \cup \Omega_2, \\ D_2 &:= \Omega_1 \cup \Omega_3. \end{aligned}$$



Image



Level set function 1



Level set function 2

Hamilton-Jacobi equation

$$\phi_t(t, x) + v_n(t, x)|\nabla\phi(t, x)| = 0, \tag{10}$$

where ϕ_t time derivative of ϕ and $\phi(0, x)$ given data.

- Note that v_n is defined only on Γ, therefore it is necessary to define its extension to the entire domain (or at least a band around the actual contour).

MS-functional

$$\mathcal{J}_\nu(\{\Omega_i\}_{i=1}^m) = \sum_{i=1}^m \int_{\Omega_i} (f(x) - c_i)^2 \, dx + \frac{\nu}{2} \sum_{i=1}^m \mathcal{H}^1(\Gamma_i) + \frac{\nu}{2} |\partial\Omega|. \tag{11}$$

For convenience, we use the notation

$$\mathcal{J}_\nu(\Gamma) = \mathcal{J}_\nu(\{\Omega_i\}_{i=1}^m).$$

Shape derivative

$$
\begin{aligned}
d\mathcal{J}_\nu(\Gamma, V) \quad = \quad & \sum_{i=1}^{m} 2 \int_{\Omega_i} (f(x) - c_i) c_i'(\Gamma, V)\, dx \\
& + \sum_{i=1}^{m} \int_{\Gamma_i} (f(x) - c_i)^2 v_{n_i}(x)\, dx \\
& + \frac{\nu}{2} \sum_{i=1}^{m} \int_{\Gamma_i} \kappa_i(x) v_{n_i}(x)\, dx,
\end{aligned}
$$

where

- κ_i is the curvature of Γ_i,
- n_i is the outer unit normal vector to Ω_i,
- $c_i'(\Gamma, V)$ is the shape derivative of c_i at Γ in the direction V.

Since $c_i'(\Gamma, V)$ is a scalar, we have

$$\sum_{i=1}^{m} \int_{\Omega_i} (f(x) - c_i) c_i'(\Gamma, V) \, dx = \sum_{i=1}^{m} c_i'(\Gamma, V) \int_{\Omega_i} (f(x) - c_i) \, dx = 0,$$

since c_i is the mean value of f over the domain Ω_i. Thus we obtain

$$d\mathcal{J}_\nu(\Gamma, V) = \sum_{i=1}^{m} \int_{\Gamma_i} \left((f(x) - c_i)^2 + \frac{\nu}{2} \kappa_i(x) \right) v_{n_i}(x) \, dx.$$

Therefore we have identified the shape gradient

$$\nabla \mathcal{J}_\nu(x) = (f(x) - c_i)^2 + \frac{\nu}{2} \kappa_i(x) \qquad \text{f.a.a } x \in \Gamma_i$$

and we may choose

$$v_{n_i}(x) = -(f(x) - c_i)^2 - \frac{\nu}{2} \kappa_i(x) \qquad \text{f.a.a } x \in \Gamma_i.$$

We deduce the value of the velocity on the boundaries of D_1 and D_2

$$\begin{aligned} v_n(x) &= -(f(x)-c_{12}(x))^2 \\ &\quad +(f(x)-c_{34}(x))^2 - \nu\kappa_{D_1}(x) \qquad \text{f.a.a. } x\in\partial D_1, \\ v_n(x) &= -(f(x)-c_{13}(x))^2 \\ &\quad +(f(x)-c_{24}(x))^2 - \nu\kappa_{D_2}(x) \qquad \text{f.a.a. } x\in\partial D_2, \end{aligned}$$

where c_{ij} is the piecewise constant function

$$\begin{aligned} c_{ij}(x) &= c_i \text{ if } x\in\Gamma_i, \\ c_{ij}(x) &= c_j \text{ if } x\in\Gamma_j, \end{aligned}$$

and $\kappa_{D_1},\ \kappa_{D_2}$ are the curvatures of D_1 and D_2, respectively.

Phase II - Algorithm

step 1 Initially choose ϕ_1^0 and ϕ_2^0 as signed distances to $\Omega_1^0 \cup \Omega_2^0$ and $\Omega_1^0 \cup \Omega_3^0$ where $\Omega_i^0, i \in \{1, \dots m\}$ come from phase I; set $k = 0$.

step 2 Compute normal velocities $v_{n,1}^k$ and $v_{n,2}^k$ for ϕ_1^k and ϕ_2^k. If $\|v_{n,1}^k\| = 0$ and $\|v_{n,2}^k\| = 0$ then stop; otherwise continue with 3.

step 3 Extend $v_{n,1}^k$ and $v_{n,2}^k$ to Ω. Update ϕ_1^k and ϕ_2^k by a time step in Hamilton-Jacobi equation.

step 4 Update domains $\Omega_i^k, i \in \{1, \dots m\}$ and put $k = k + 1$. Go to step 2.

Table 1

n^2	It. topo	Time topo	It. shape	Time shape
390^2	11	$0.11s$	6	$11.28s$

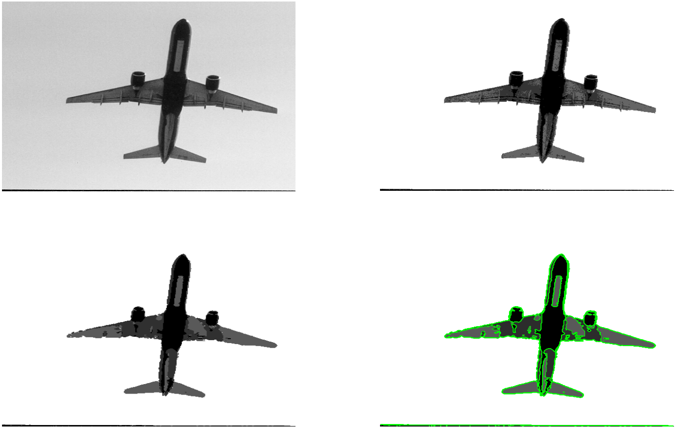


Figure: Original (ul), image after topo. step (ur), segmentation (ll) / with contour in green (lr).

size	It. topo	Time topo	It. shape	Time shape
315×315	14	$0.09s$	3	$4.05s$

Figure: Original image (upper left), image after topology step (upper right), segmentation without contour (lower left), segmentation with contour (lower right)

size	It. topo	Time topo	It. shape	Time shape
331×331	35	0.26	6	$17.83s$

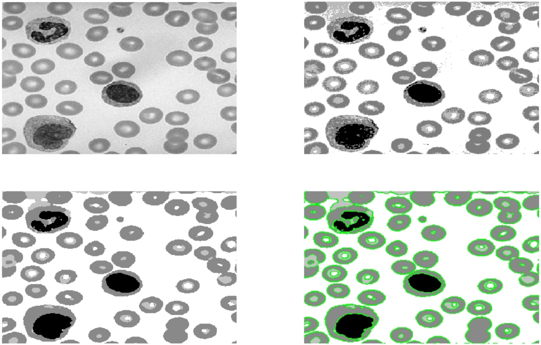


Figure: Original image (upper left), image after topology step (upper right), segmentation without contour (lower left), segmentation with contour (lower right)

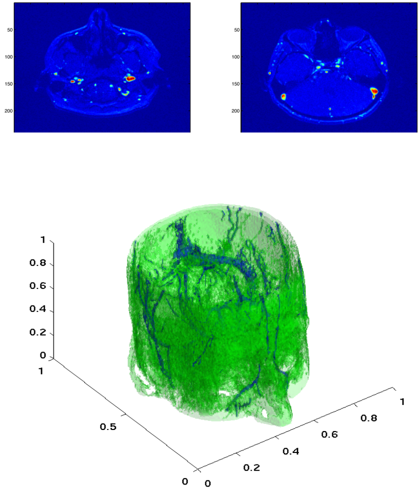
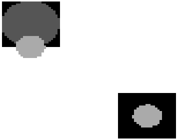


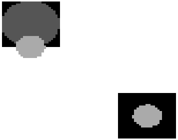
Figure: 2D slices of original (upper) and after topology optim. (lower).

size	It. topo	Time topo
$256 \times 208 \times 70$	30	4.93

The Osher-He Experience



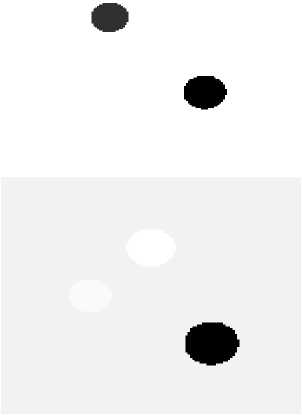
Original



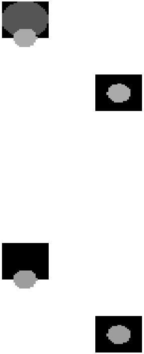
Our result

The Osher-He Experience

Initialization



Result by Osher / He

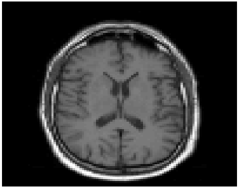


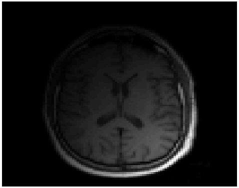
Segmentation AND modulation recovery

- Surface coils

exhibit high sensitivity σ near center of coil which falls off away.

- Modulated image





Segmentation AND modulation recovery

- We require $u \in [0,1]$ and $0 \leq \sigma \leq \bar{\sigma}$ in Ω.
- This motivates the following approximation:

$$
\begin{aligned}
J_\nu(u,\Gamma,\sigma) &= \int_\Omega (\sigma^{-1}f-u)^2 + \delta\int_\Omega |\nabla^2\sigma|^2 + \mu\int_{\Omega\setminus\Gamma}|\nabla u|^2 + \nu\mathcal{H}^1(\Gamma) \\
&\quad + \kappa\int_\Omega \max(u-1,0)^2 - \lambda\int_\Omega (\ln(\sigma)+\ln(\bar{\sigma}-\sigma)),
\end{aligned}
$$

  with $\kappa > 0$ and $\lambda > 0$ (driven to 0 over iterations).
- Solution process: nonlinear Gauss-Seidel, i.e., minimize w.r.t. one variable while keeping the respective other one fixed.
  - For u fixed: Newton-Multigrid solver for fourth order PDE

$$
\begin{aligned}
(\delta\Delta^2 + u^2)\sigma - \frac{\lambda}{2\sigma} + \frac{\lambda}{2(\bar{\sigma}-\sigma)} &= uf \quad \text{in } \Omega, \\
\partial_{nn}\sigma = \partial_{n\tau}\sigma = \partial_n\Delta\sigma &= 0 \quad \text{on } \Gamma,
\end{aligned}
$$

Segmentation AND modulation recovery

- For σ fixed: TOPSHAPE with necessary optimality conditions for c_i, $i \in \{1, .., m\}$, given by

$$2\kappa|\Omega_i| \max(c_i - 1, 0) + \int_{\Omega_i} 2\left(c_i - \frac{f}{\sigma}\right) = 0.$$

  This leads to the two cases

$$c_i = |\Omega_i|^{-1} \int_{\Omega_i} \frac{f}{\sigma} \text{ if } c_i \leq 1,$$

  and

$$c_i = \frac{\kappa + |\Omega_i|^{-1} \int_{\Omega_i} \frac{f}{\sigma}}{1 + \kappa} \text{ if } c_i > 1.$$

  Note that in both situations $c_i \geq 0$ for all i.

Segmentation AND modulation recovery

- If $|\Omega_j| \neq 0$, we get

$$\begin{aligned}\mathcal{T}_{ij}(x_0) \quad &= \quad \left(\frac{f(x_0)}{\sigma(x_0)} - c_j\right)\left(\frac{f(x_0)}{\sigma(x_0)} - c_j - \max(c_j - 1, 0)\frac{2\kappa|\Omega_j|}{(\kappa + |\Omega_j|)^2}\right) - \\ &\quad \left(\frac{f(x_0)}{\sigma(x_0)} - c_i\right)\left(\frac{f(x_0)}{\sigma(x_0)} - c_i - \max(c_i - 1, 0)\frac{2\kappa|\Omega_i|}{(\kappa + |\Omega_i|)^2}\right).\end{aligned}$$

When $|\Omega_j| = 0$ we obtain

$$\mathcal{T}_{ij}(x_0) \quad = \quad -\left(\frac{f(x_0)}{\sigma(x_0)} - c_i\right)\left(\frac{f(x_0)}{\sigma(x_0)} - c_i - \max(c_i - 1, 0)\frac{2\kappa|\Omega_i|}{(\kappa + |\Omega_i|)^2}\right).$$

Segmentation AND modulation recovery