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Surprises at the classical limit.

Classical mechanics appears at the "boundary" of quantum mechanics, and this boundary is reached when a certain combination of parameters of the system under consideration, an "effective Planck constant", tends to zero. There is only one world, the quantum world we live in, and "classical effects" are visible only when the Planck constant is very small. This is the case for macroscopic systems and that is the reason why these effects are familiar to us. Traditionally, it is understood that the classical world fills up this border and that quantum mechanics reduces to classical mechanics when the Planck constant vanishes.
But this transition, compared to other asymptotics, when typical speeds are small compared to the speed of light for example, is not direct and aims at the link between two very different mathematical paradigms: the Schrödinger equation, a partial differential equation, in the quantum world, and the Hamilton equations, a system of ordinary differential equations in the classical case. In fact, one can discover a much more complex and richer "border structure", containing, but not only, the classical paradigm. This will be the main subject of this course.

The structure of the course will be as follows. We will begin with a review of the basic techniques and results concerning elementary quantum mechanics, the Schrödinger equation and Hamiltonian systems, and will then present the traditional "semiclassical approximation", where one assumes that the data (potentials and initial conditions) are, say, "smooth" and that the limit is reached in finite time. We will show how this construction not only leads to, but actually defines classical mechanics as a part of the quantum world. The larger part of the lectures will then be dedicated mostly to important examples of "non-standard" classical limits. We will cover three natural situations where such effects show up:

a) the time evolution diverges as the Planck constant vanishes which leads to new approaches of infinite time classical evolution;

b) the potentials are not smooth enough for the Hamilton flow to be well defined (the Cauchy-Lipschitz condition is not satisfied);

c) the initial data are not smooth such that semiclassical evolution gives rise to phenomena of concentration on Cantor sets, for example.

The prerequisites for the course are rather elementary : basic functional analysis (Hilbert spaces, self-adjointness, Stone theorem), elementary measure theory (Lebesgue measure, Lebesgue continuity, Radon-Nikodym theorem) and basics in differential equations (local existence, unicity, Cauchy-Lipschitz theorem).

More advanced references are (but all the needed material will be presented during the lectures):

- T. Paul, "Semiclassical methods with an emphasis on coherent states", Tutorial Lectures, Proceedings of the conference "Quasiclassical methods", B. Simon (et/and) J. Rauch, eds., IMA Series, Springer Verlag 1997.

- P.L. Lions and T. Paul, "Sur les fonctions de Wigner", Revista Mat. Ibero. Vol 9, 553-618, 1994.

- L. Ambrosio, "Transport equation and Cauchy problem for non-smooth vector". Lecture Notes in Mathematics "Calculus of Variations and Non-Linear Partial Differential Equations"(CIME Series, Cetraro, 2005) 1927, B. Dacorogna, P. Marcellini eds., 2-41, 2008.