BMS Basic Course
Commutative Algebra

Jürg Kramer

26.10.2020

Formalities

- Lecturer: Jürg Kramer
- Assistant: Thorsten Herrig
- Format: Online
- Time of Lectures: Tuesday: 09:15 am – 10:45 am
  Thursday: 09:15 am – 10:45 am
- Time of Tutorials: Thursday: 11:15 am – 12:45 am
- First Lecture: Tuesday, November 3rd, 2020
- Webpage: www.math.hu-berlin.de/∼kramer

Moodle Course

All the materials provided and needed for the Basic Course "Commutative Algebra", like

- zoom links for lectures and tutorials,
- slides of the lectures,
- new exercise sheets,
- literature, etc.

are to be found in the respective Moodle Course, which can be accessed via

- link: https://moodle.hu-berlin.de/course/view.php?id=98627
- password: Hilbert2021

Solutions of the exercises and the respective corrections are also uploaded via this Moodle Course.

One motivation comes from number theory

The fundamental theorem of arithmetic states that every non-zero integer can be uniquely (up to order) decomposed as a product of powers of prime numbers.

Passing to domains generalizing the ring of integers like the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b \cdot \sqrt{-5} \mid a, b \in \mathbb{Z}\},$$

we find that the fundamental theorem in the above form fails to hold. Namely, in the ring $\mathbb{Z}[\sqrt{-5}]$, we have the two essentially different factorizations

$$6 = 2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}),$$

where the numbers $2, 3$, and $1 \pm \sqrt{-5}$ cannot be further decomposed in $\mathbb{Z}[\sqrt{-5}]$.

The crucial idea is to replace "numbers" by so-called **ideals**, in particular "prime numbers" by so-called **prime ideals**, and then show as a replacement of the fundamental theorem of arithmetic:

Corollary to the theorem about the existence and the uniqueness of primary decompositions:

Every non-zero ideal in a Dedekind domain can be uniquely (up to order) decomposed as a product of powers of prime ideals.

Another motivation comes from algebraic geometry

The fundamental theorem of algebra states that every non-constant polynomial in one variable with complex coefficients has all its roots in the field of complex numbers.

Generalization

Let $P_j(X_1, \dots, X_n) \in \mathbb{C}[X_1, \dots, X_n]$ with $j \in \{1, \dots, m\}$ be m non-constant polynomials in the n variables $X_1, \dots, X_n$ with complex coefficients. Let then $V(P_1, \dots, P_m) \subseteq \mathbb{C}^n$ denote the set of all common zeroes of $P_1, \dots, P_m$ in $\mathbb{C}^n$, i.e.,

$$V(P_1, \dots, P_m) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid P_j(x_1, \dots, x_n) = 0, \; \forall j\}.$$

Question: $V(P_1, \dots, P_m) \neq \emptyset$?

Hilbert's Nullstellensatz

If the ideal generated by the polynomials $P_1, \ldots, P_m$ is a **proper** ideal in the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$, then the set of common zeroes $V(P_1, \ldots, P_m) \subseteq \mathbb{C}^n$ is not the empty set.

Planned Contents of the Course

- Rings, ideals, and modules
- Prime, maximal, and primary ideals
- Elements of homological algebra
- Primary decompositions
- Noetherian and Artinian rings
- Discrete valuation rings and Dedekind domains
- Classification of semisimple algebras

Thank you for your attention!