Groupoids and applications in mathematical physics

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April 25th - July 25th, Lectures: Mondays 12:15pm-1:45pm, Building 9, Golm Campus 2.09.1.10
April 27th - July 27th, Exercise sessions: Wednesdays 12:15pm-1:45pm, Building 25, Golm Campus 2.25.D1.02

Abstract

Groupoids, which generalise fibrations of groups over manifolds, have become central tools to host singular structures. After some prerequisites on groupoids, we shall present three instances in which groupoids arise in different forms and for different purposes. We shall first discuss van Erp and Yuncken's groupoid approach to pseudodifferential operators by means of the tangent groupoid. We shall then define and study direct connections on groupoids and their jet prolongations, which play a role in Hairer's regularity structures. If time allows, we shall finally discuss principal bundles groupoids, which when viewed as groupoids in the category of principal bundles, can be seen as building blocks for higher gauge theories.

Introduction

Groupoids first introduced by Brandt[1], generalise fibrations of groups over manifolds. Interest in groupoids broadened in the 50's when the notion of category arose, since the invertible elements of a small category form a groupoid. As from then, the use of groupoids was expanded by Ehresmann in various areas of mathematics, including differential geometry. Groupoids have become central tools to host singular structures. After some prerequisites on groupoids, this course presents three instances in which groupoids arise in various forms and for different purposes.

1. **Prerequisites on groupoids.** We first review basic definitions relative to groupoids, including algebroids, gauge groupoids and the tangent groupoid, which play an important role in the sequel. We shall mostly follow [MK05].

2. **A groupoid approach to pseudodifferential operators.** We then recall basic concepts of pseudodifferential operators on smooth manifolds. We further discuss van Erp and Yuncken's [vEY17a] groupoid approach to pseudodifferential operators which uses the Debord-Skandalis zoom action on the tangent groupoid. We finally compare their approach with that of Nistor, Weinstein and Xu [NWX99]. This chapter partially follows the presentation of [L21].

3. **Direct connections.** We further discuss the concept of direct connection, which arises under various disguises in algebraic, geometric and analytic contexts. They were first introduced by Teleman [Te04, Te07, KT06] under the name linear direct connection in the context of non commutative geometry, where they live on frame groupoids. Direct connections also arise as re-expansion (or transport) maps in Hairer's regularity structures [H14, H2] on an Euclidean vector space which were later generalised to a Riemannian manifold in [DDD19]. Here again they arise in the context of singularities, since regularity structures offer an algebraico-analytic device to transform a singular stochastic differential equation into a fixed point problem. We shall discuss their infinitesimal connections, their curvature and their extensions to jet groupoids. This chapter is based on [ABFP22].

4. **Principal bundle groupoids and bundle gerbes.** We finally discuss principal bundles groupoids, which when viewed as groupoids in the category of principal bundles, can be seen as building blocks for higher gauge theories [BH11]. Their structure group is a Lie group groupoid, which can also be viewed as a Lie 2-group or a crossed module built from the action of a Lie group on another Lie group. We build their adjoint bundle groupoids, which we interpret as vector bundle groupoids, widely discussed in the literature. Inspired by [NW13], we show how from a principal bundle groupoid whose 0 level is a trivial principal bundle gives rise to a bundle gerbe and conversely. This chapter is based on ongoing joint work with Alfonso Garmendia.

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[1]Brandt actually introduced in [Bra26] what are now called transitive groupoids.

1 Prerequisites on groupoids

1.1 Groupoids and groupoid morphisms

1.2 Action groupoids

1.3 Lie algebroids

1.4 The exponential map and the tangent groupoid

2 Pseudodifferential operators: a groupoid perspective

2.1 Pseudodifferential operators on sections of a vector bundle

2.1.1 Pseudodifferential symbols

2.1.2 Pseudodifferential operators and their distribution kernels

2.2 Pseudodifferential operators on groupoids

2.2.1 Distributions on Lie groupoids

2.2.2 Van Erp-and Yuncken's tangent groupoid approach

2.2.3 $\mathcal{G}$- pseudodifferential operators following Nistor, Weinstein and Xu

3 Direct connections on groupoids

3.1 Direct connections

3.2 Jet groupoids

3.3 Direct connections on jet groupoids

3.4 Direct connections in regularity structures

4 Principal bundle groupoids versus bundle gerbes and higher gauge theory[NW13]

4.1 Principal bundle groupoids (PBG)

4.2 Bundle gerbes

4.3 From PBGs to bundle gerbes and back

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