Imaging Science meets Compressed Sensing

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BMS Friday Colloquium
January 6, 2012

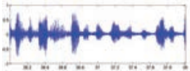
Outline

General Challenge in Data Analysis

Modern Data in general is often composed of two or more morphologically distinct constituents, and we face the task of separating those components given the composed data.

Examples include...

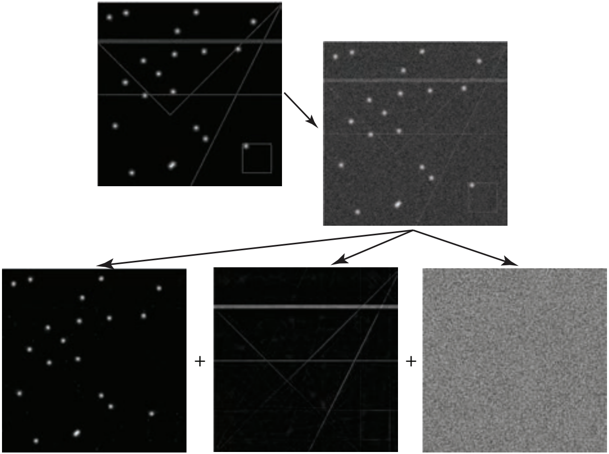
- Audio data: Different instruments.



- Imaging data: Cartoon and texture.

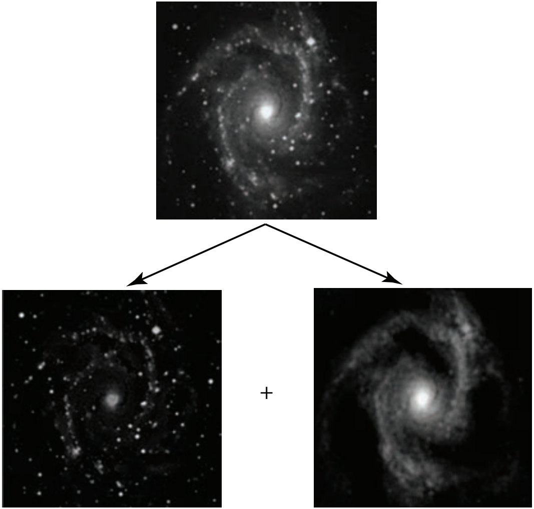
- High-dimensional data: Lower-dimensional structures of different dimensions.

Separating Artifacts in Images, I



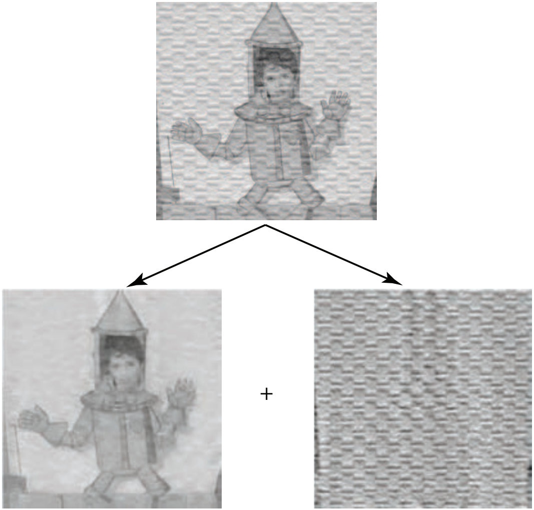
(Source: J. L. Starck, M. Elad, D. L. Donoho; 2005 (Artificial Data))

Separating Artifacts in Images, II



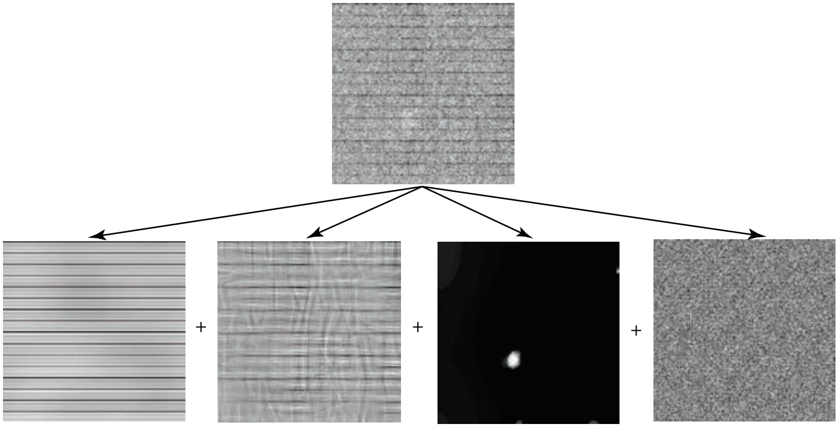
(Source: J. L. Starck, M. Elad, D. L. Donoho; 2006)

Separating Artifacts in Images, III



(Source: J. L. Starck, M. Elad, D. L. Donoho; 2006)

Separating Artifacts in Images, IV

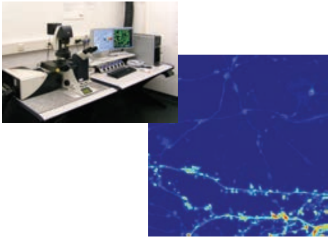


(Source: J. L. Starck, M. Elad, D. L. Donoho; 2005)

Problem from Neurobiology

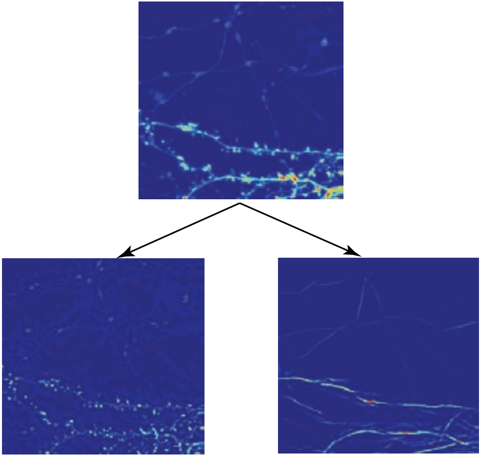
Alzheimer Research:

- Detection of characteristics of Alzheimer.
- Separation of spines and dendrites.



(Confocal-Laser Scanning-Microscopy)

Numerical Result



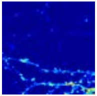
(Source: Brandt, K, Lim, Sündermann; 2010)

Goal for Today

Neurobiological Data:

Observed signal $x = x_1 + x_2$.

- $x_1 =$ Point structures.
- $x_2 =$ Curvilinear structures.



Challenges for Today:

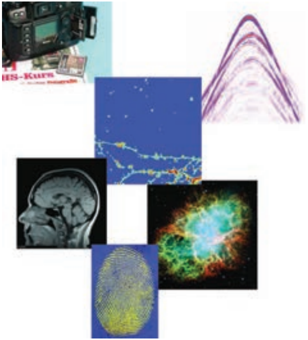
- Mathematical methodology to derive the empirical results!
- Fundamental mathematical concept behind the empirical results!

What is

Modern Imaging Science?

Numerous Tasks in Imaging Science

- Denoising.
- Deblurring.
- Inpainting.
- Component Separation.
- Superresolution.
- ...



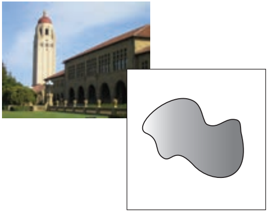
Examples for Modeling of Image Data

Digital Model:

- $A \in \mathbb{R}^{N \times N}$.

Continuum Model:

- $f \in L^2([0,1]^2)$.
- $f \in \mathcal{D}'(\mathbb{R}^2)$.
- ...



$\rightsquigarrow$ **What is a ‘natural’ image?**

Applied Harmonic Analysis Approach to Imaging Science

Exploit a carefully designed representation system $(\psi_\lambda)_{\lambda\in\Lambda} \subseteq L^2(\mathbb{R}^2)$:

$$L^2(\mathbb{R}^2) \ni f \longrightarrow (\langle f, \psi_\lambda\rangle)_{\lambda\in\Lambda} \longrightarrow \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \, \psi_\lambda = f.$$

Desiderata:

- Special features encoded in the "large" coefficients $|\langle f, \psi_\lambda\rangle|$.
- Efficient representations:

$$f \approx \sum_{\lambda\in\Lambda_N} \langle f, \psi_\lambda\rangle \, \psi_\lambda, \quad \#(\Lambda_N) \text{ small}$$

Methodology:

- Modification of the coefficients according to the task.

Other Approaches to Imaging Science

PDE-based Methods:

- Given an image $f \in L^2(\mathbb{R}^2)$.
- Let $g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, $g(0, x) = f(x)$.
- Solve

$$F(t, x, g, \partial_1 g, ...) = 0, \quad g(0, x) = f(x).$$

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Variational Methods:

- Given an image $f \in L^2(\mathbb{R}^2)$.
- Introduce functionals $\Phi, \Psi : L^2(\mathbb{R}^2) \to \mathbb{R}$.
- Solve

$$\min_g \Phi(f - g) + \mu \Psi(g).$$

How does Compressed Sensing help with Component Separation?

'Mathematical Model'

Model for 2 Components:

- Observe a signal x composed of two subsignals x_1 and x_2:

$$x = x_1 + x_2.$$

- Extract the two subsignals x_1 and x_2 from x, if only x is known.

‘Mathematical Model’

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- There are two unknowns for every datum.

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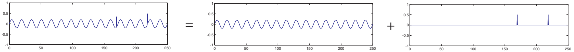
But we have additional Information:

- The two components are geometrically different.

Birth of Component Separation using Compressed Sensing

Problem:

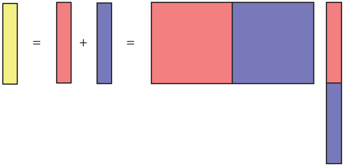


Composition of Sinusoids and Spikes sampled at n points:

$$x = x_1^0 + x_2^0 = \Phi_1 c_1^0 + \Phi_2 c_2^0 = \left[\quad \Phi_1 \quad \middle| \quad \Phi_2 \quad \right] \begin{bmatrix} c_1^0 \\ c_2^0 \end{bmatrix},$$

where

- x, c_1^0, and $c_2^0 \in \mathbb{R}^n$.
- Φ_1 is the $n \times n$-Fourier matrix ($(\Phi_1)_{t,k} = e^{2\pi i t k/n}$).
- Φ_2 is the $n \times n$-Identity matrix.

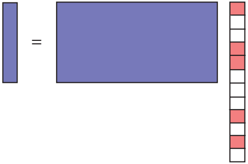


Compressed Sensing

Observation:

Let A be an $n \times N$-matrix, $n << N$. In many situations the seeked solution c^0 of $x = Ac^0$ is sparse, i.e.,

$$\|c^0\|_0 = \#\{i : c_i^0 \neq 0\} \text{ is 'small'.}$$



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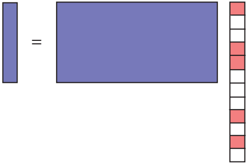

First idea: Solve...

$$(P_0) \quad \min_c \|c\|_0 \text{ such that } x = Ac$$

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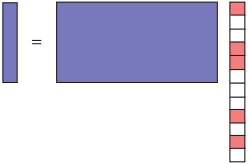

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$\rightsquigarrow$ This problem is NP-hart!

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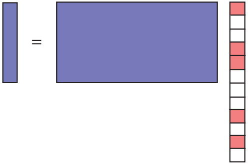

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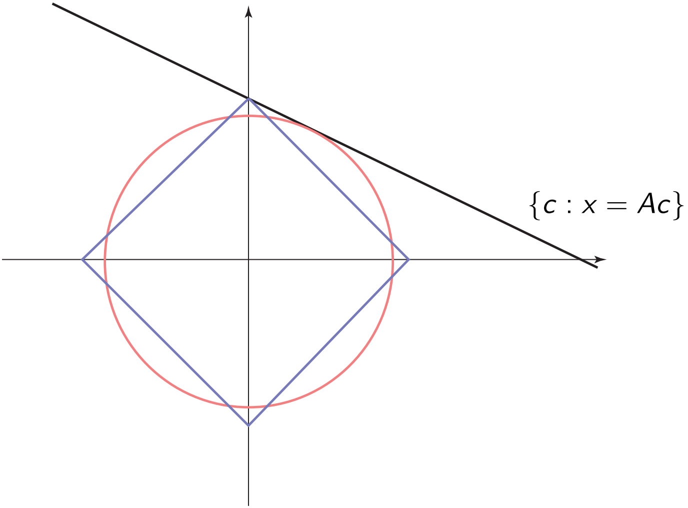
$\leadsto$ This problem is NP-hart!

Basis Pursuit (Chen, Donoho, Saunders; 1998)

$$(P_1) \quad \min_c \|c\|_1 \text{ such that } x = Ac$$

Intuition



Exact Recovery by ℓ_1 Minimization

Meta-Result: If

- $\|c^0\|_0$ is sufficiently small,
- A is sufficiently incoherent,

then

$$c^0 = \text{argmin}_c \|c\|_1 \text{ such that } x = Ac.$$

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Exemplary Result (Donoho, Elad; 2003)

Let A be an $n \times N$-matrix with normalized columns, $n << N$, and let $c^0 \in \mathbb{R}^N$ satisfy

$$\|c^0\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right),$$

where the coherence $\mu(A)$ is defined by $\mu(A) = \max_{i \neq j} |\langle a_i, a_j \rangle|$.
Then

$$c^0 = \text{argmin}_c \|c\|_1 \text{ such that } x = Ac.$$

Birth of Component Separation using Compressed Sensing

Composition of Sinusoids and Spikes sampled at n points:

$$x = x_1^0 + x_2^0 = \Phi_1 c_1^0 + \Phi_2 c_2^0 = \left[\quad \Phi_1 \quad \middle| \quad \Phi_2 \quad \right] \left[\begin{array}{c} c_1^0 \\ c_2^0 \end{array} \right].$$

Coherence of $[\Phi_1|\Phi_2]$:

$$\mu([\Phi_1|\Phi_2]) = \mu([F|I]) = \frac{1}{\sqrt{n}}.$$

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Theorem (Donoho, Huo; 2001)

If $\#(\text{Sinusoids}) + \#(\text{Spikes}) = \|(c_1^0)\|_0 + \|(c_2^0)\|_0 < (1+\sqrt{n})/2$, then

$$(c_1^0, c_2^0) = \operatorname{argmin}(\|c_1\|_1 + \|c_2\|_1) \text{ subject to } x = \Phi_1 c_1 + \Phi_2 c_2.$$

Component Separation using Compressed Sensing

Let x be a signal composed of two subsignals x_1^0 and x_2^0:

$$x = x_1^0 + x_2^0.$$

Desiderata for two orthonormal bases Φ_1 and Φ_2:

- $x_i^0 = \Phi_i c_i^0$ with $\|c_i^0\|_0$ small, $i = 1, 2$ $\rightsquigarrow$ Sparsity!
- $\mu([\Phi_1|\Phi_2])$ small $\rightsquigarrow$ Morphological Difference!

Solve

$$(c_1^*, c_2^*) = \text{argmin}(\|c_1\|_1 + \|c_2\|_1) \;\text{ subject to }\; x = \Phi_1 c_1 + \Phi_2 c_2$$

and derive the approximate components

$$x_i^0 \approx x_i^* = \Phi_i c_i^*, \quad i = 1, 2.$$

Two Paths

Avalanche of Recent Work

Problem: Solve $x = Ac^0$ with A an $n \times N$-matrix ($n < N$).

Results using structured matrices A:

- A is often to some extent given by the application.
- When can c^0 still be recovered and how fast?
- Contributors: *Candès, Donoho, Elad, Rauhut, Temlyakov, Tropp, ...*

Results using random matrices A:

- The 'best' A is a random matrix.
- What is maximally possible if A can be freely chosen?
- Contributors: *Candès, Donoho, Pajor, Romberg, Tanner, Tao, ...*

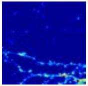
Remark: Matheon-Talk by Emmanuel Candès (June 20th).

How can these Ideas be applied to

Separation of Points and Curves?

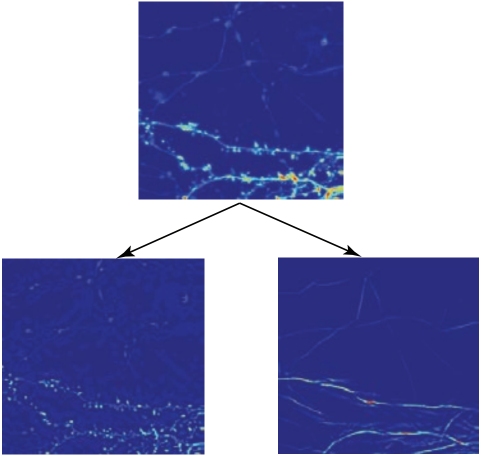
Back to Neurobiological Imaging

- Two morphologically distinct components:
  - Points
  - Curves



- Choose suitable representation systems which provide optimally sparse representations of
  - pointlike structures $\longrightarrow$ Wavelets
  - curvelike structures $\longrightarrow$ Shearlets
- Minimize the ℓ_1 norm of the coefficients.
- This forces
  - the pointlike objects into the wavelet part of the expansion
  - the curvelike objects into the shearlet part.

Empirical Separation of Spines and Dendrites



Wavelet Expansion

Shearlet Expansion

(Source: Brandt, K, Lim, Sündermann; 2010)

Wavelets

Definition:
The wavelet system associated with $\psi \in L^2(\mathbb{R}^2)$ is defined by

$$\{\psi_{j,m}(x) = 2^j \psi(\begin{pmatrix} 2^j & 0 \\ 0 & 2^j \end{pmatrix} x - m) \, : \, j \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Theorem:
Let $f \in C^2(\mathbb{R}^2)$ except finitely many point singularities. Then wavelets provide an optimally sparse approximation of f, i.e.,

$$\|f - f_N\|_2^2 \leq C \cdot N^{-1}, \quad N \to \infty, \quad \text{where } f_N = \sum_{\lambda \in \Lambda_N} c_\lambda \psi_\lambda.$$

Beyond Wavelets...

Observation:

- Wavelets can not approximate curvilinear singularities optimally sparse.
- Reason: Isotropic structure of wavelets:

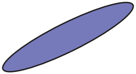
$$2^j \psi(\begin{pmatrix} 2^j & 0 \\ 0 & 2^j \end{pmatrix} x - m)$$

Intuitive explanation:

Shearlets

Parabolic scaling:

$$A_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad j \in \mathbb{Z}.$$



Orientation via shearing:

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Definition (K, Labate, Lim; 2006):
For $\psi \in L^2(\mathbb{R}^2)$, the associated shearlet system is defined by

$$\mathcal{SH}(\psi) = \{2^{\frac{3j}{4}}\psi(S_k A_j \cdot -m) \,:\, j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

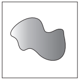
Compactly Supported Shearlets

Theorem (Kittipoom, K, Lim; 2010):
Let $\psi \in L^2(\mathbb{R}^2)$ be compactly supported, and let $\hat{\psi}$ satisfy certain decay conditions. Then $\mathcal{SH}(\psi) = (\sigma_\eta)_\eta$ forms a frame with controllable frame bounds, i.e.,

$$A\|f\|_2^2 \leq \sum_\eta |\langle f, \sigma_\eta \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}^2).$$

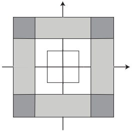
Theorem (K, Lim; 2010):
Let $\psi \in L^2(\mathbb{R}^2)$ be compactly supported, and let $\hat{\psi}$ satisfy certain decay conditions. Then $\mathcal{SH}(\psi)$ provides an optimally sparse approximation of f, i.e.,

$$\|f - f_N\|_2^2 \leq C \cdot N^{-2} \cdot (\log(N))^3, \quad N \to \infty.$$

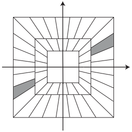


Chosen Pair

Optimal for Pointlike Structures:
Orthonormal Wavelets are a basis with perfectly isotropic generating elements at different scales.



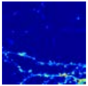
Optimal for Curvelike Structures:
Shearlets (K, Labate, Lim; 2006) are a highly directional frame with increasingly anisotropic elements at fine scales ($\longrightarrow$ www.ShearLab.org).



Separation Algorithm

Observed Object:

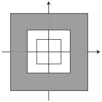
$$f = \mathcal{P}^0 + \mathcal{C}^0.$$



Subband Decomposition:

Wavelets and shearlets use the same scaling subbands!

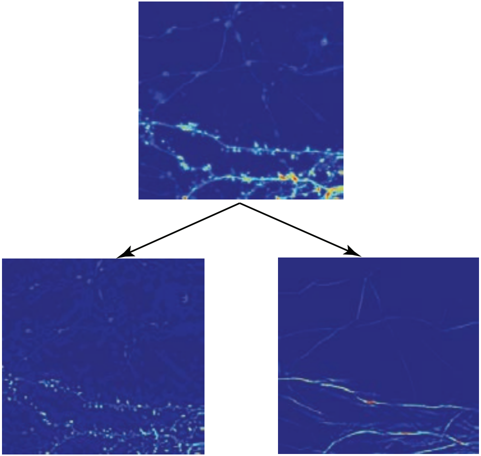
$$f_j = \mathcal{P}_j^0 + \mathcal{C}_j^0, \quad \mathcal{P}_j^0 = \mathcal{P}^0 \star F_j \text{ and } \mathcal{C}_j^0 = \mathcal{C}^0 \star F_j.$$



ℓ_1-Decomposition:

$$(\mathcal{P}_j^*, \mathcal{C}_j^*) = \text{argmin} \|(\langle \mathcal{P}_j, \psi_\lambda \rangle)_\lambda\|_1 + \|(\langle \mathcal{C}_j, \sigma_\eta \rangle)_\eta\|_1 \text{ s.t. } f_j = \mathcal{P}_j + \mathcal{C}_j$$

Empirical Separation of Spines and Dendrites

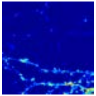


Wavelet Expansion Shearlet Expansion

(Source: Brandt, K, Lim, Sündermann; 2010)

Microlocal Model

Neurobiological Geometric Mixture in 2D:



Point Singularity:

$$\mathcal{P}^0(x) = \sum_{i=1}^{P} |x - x_i|^{-3/2}$$

Curvilinear Singularity:

$$\mathcal{C}^0 = \int \delta_{\tau(t)} dt, \quad \tau \text{ a closed } C^2\text{-curve.}$$

Observed Signal:

$$f = \mathcal{P}^0 + \mathcal{C}^0$$

Asymptotic Separation

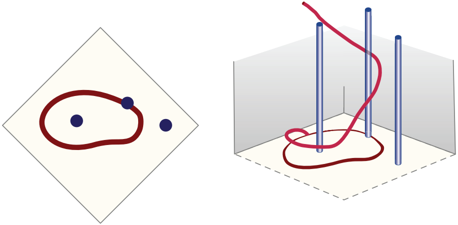
Theorem (Donoho, K; 2010)

$$\frac{\|\mathcal{P}_j^* - \mathcal{P}_j^0\|_2 + \|\mathcal{C}_j^* - \mathcal{C}_j^0\|_2}{\|\mathcal{P}_j^0\|_2 + \|\mathcal{C}_j^0\|_2} \to 0, \qquad j \to \infty.$$

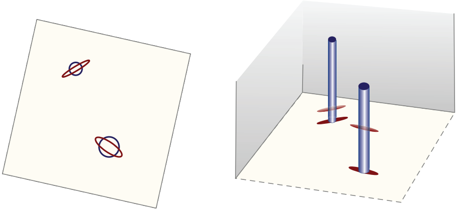
At all sufficiently fine scales, nearly-perfect separation is achieved!

Microlocal Analysis Heuristics

Singular Support and Wavefront Set of $\mathcal{P}^0$ and $\mathcal{C}^0$:



Phase Space Portrait of Wavelets and Shearlets:



Let's conclude...

What to take Home...?

- One main task in imaging science: Component Separation.
- One approach to imaging science: Applied Harmonic Analysis.
- Compressed Sensing allows exact solution of underdetermined linear systems of equations if the solution is sparse and the matrix is incoherent.
- Separation of point- and curvelike structures:
  - Wavelets sparsify points and shearlets sparsify curves.
  - Morphological distance encoded in incoherence.
  - Solution: ℓ_1 minimization.

THANK YOU!

References available at:

`page.math.tu-berlin.de/∼kutyniok`

Related Books:

- Y. Eldar and G. Kutyniok
  Compressed Sensing: Theory and Applications
  Cambridge University Press, 2012.

- G. Kutyniok and D. Labate
  Shearlets: Multiscale Analysis for Multivariate Data
  Birkhäuser-Springer, 2012.