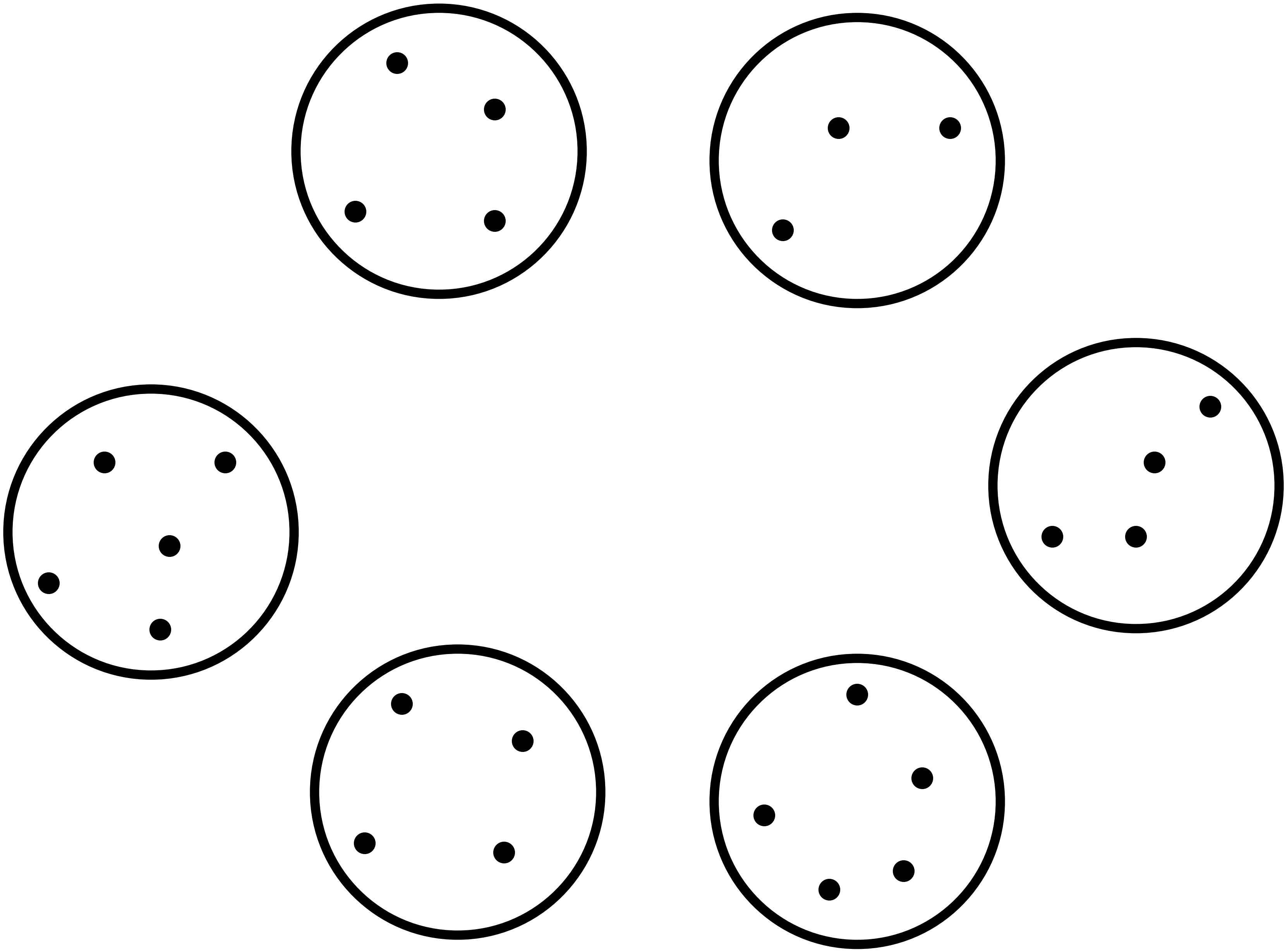
Independent Transversals

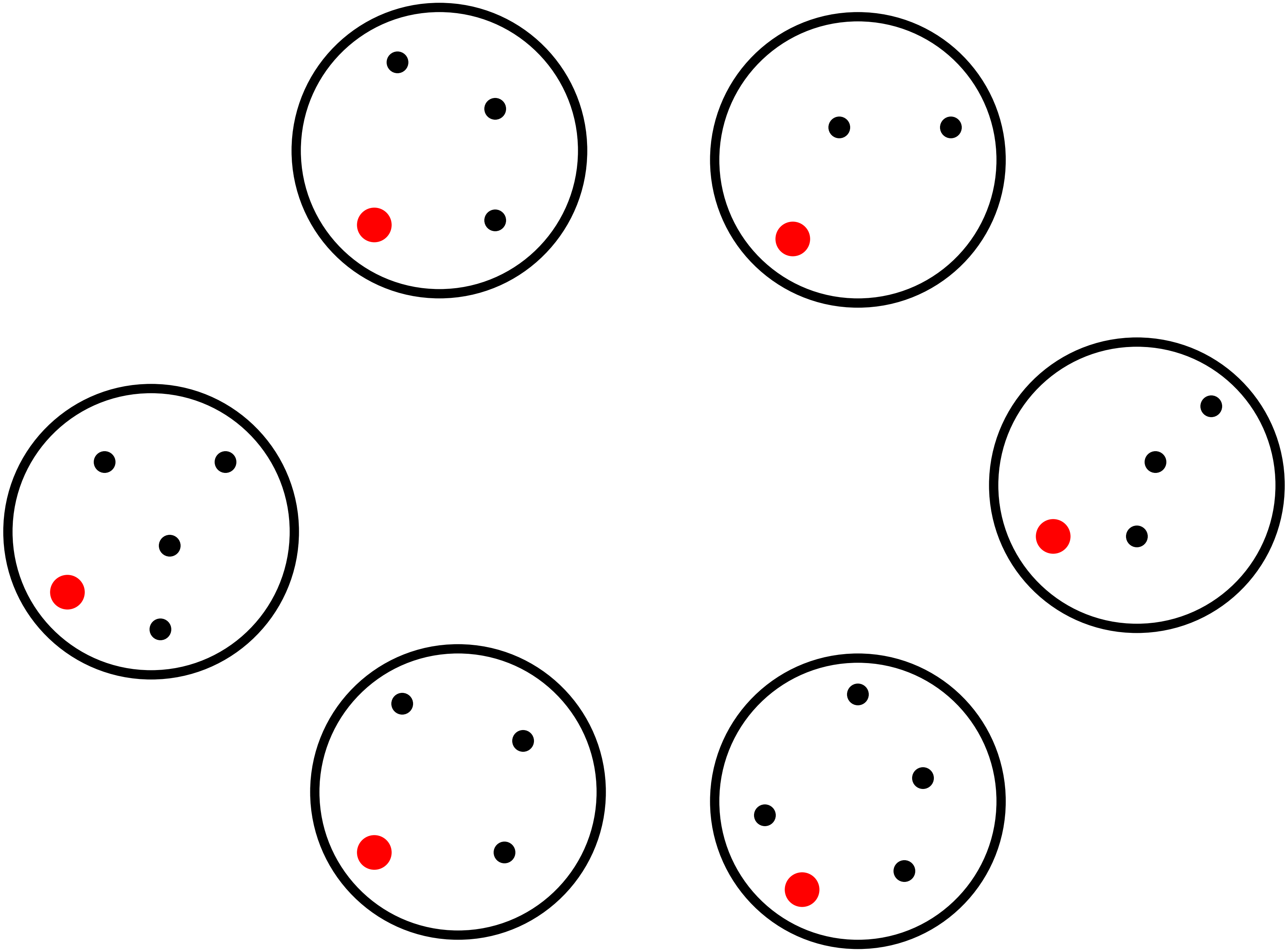
or

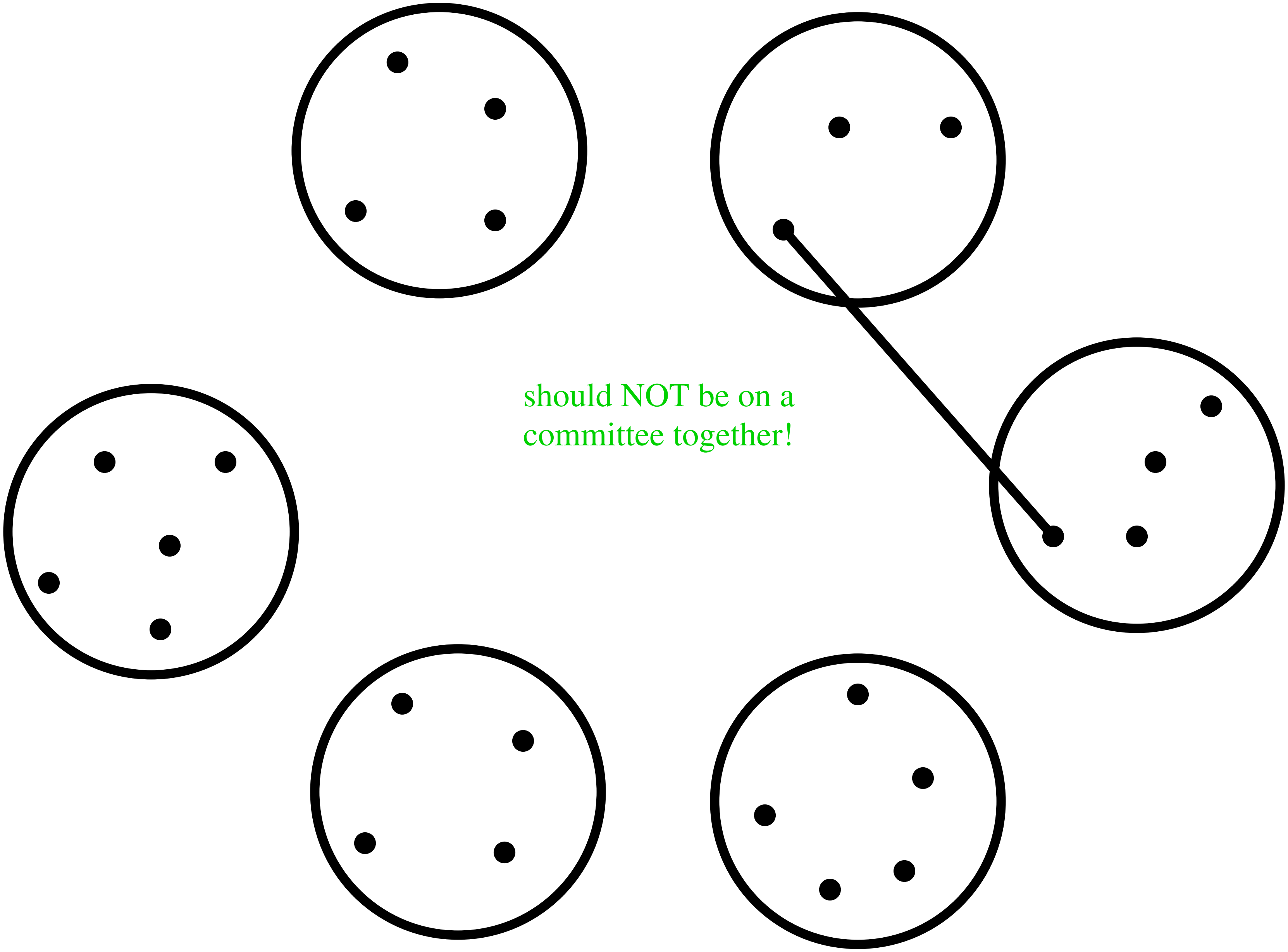
on Forming Committees

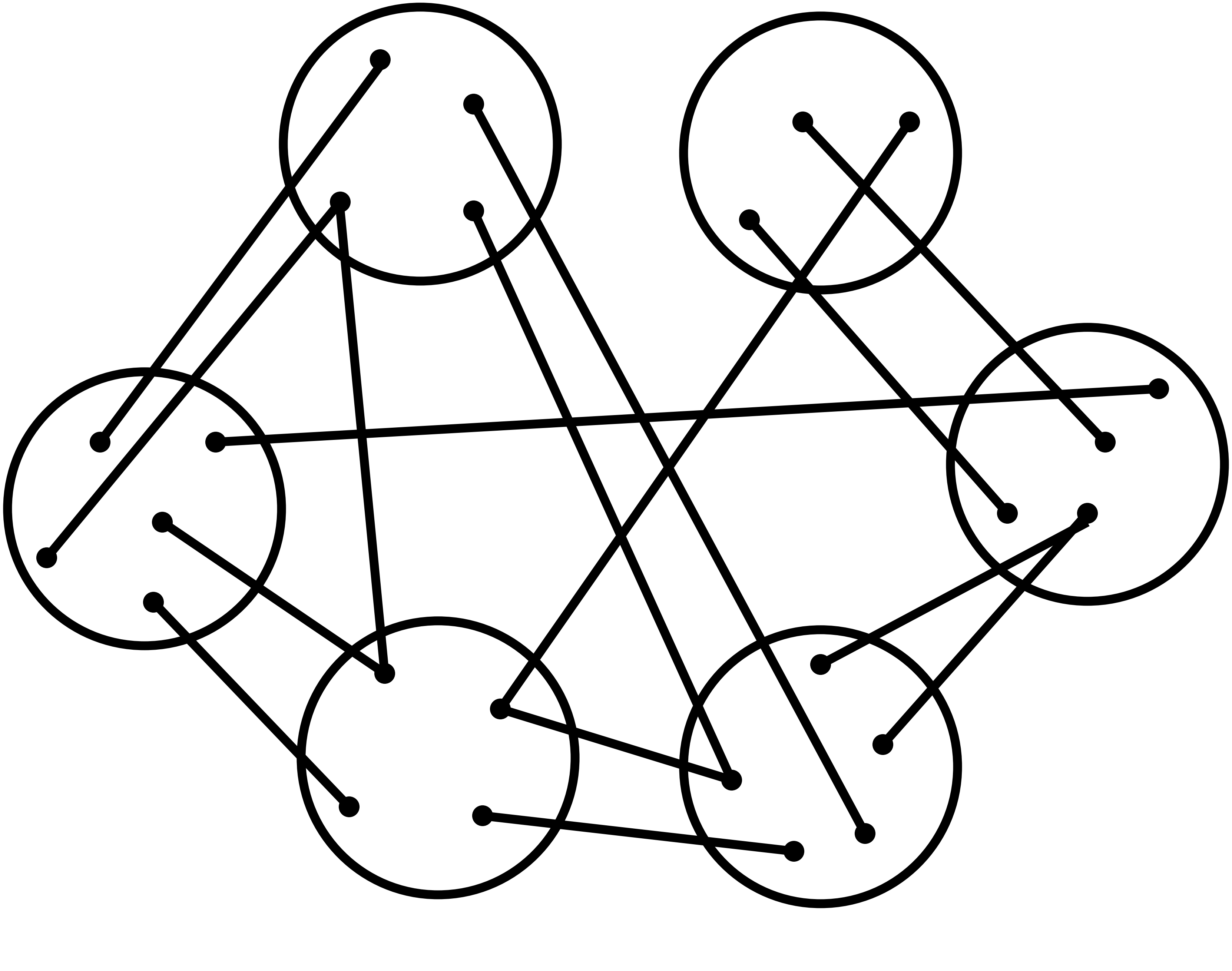
Penny Haxell

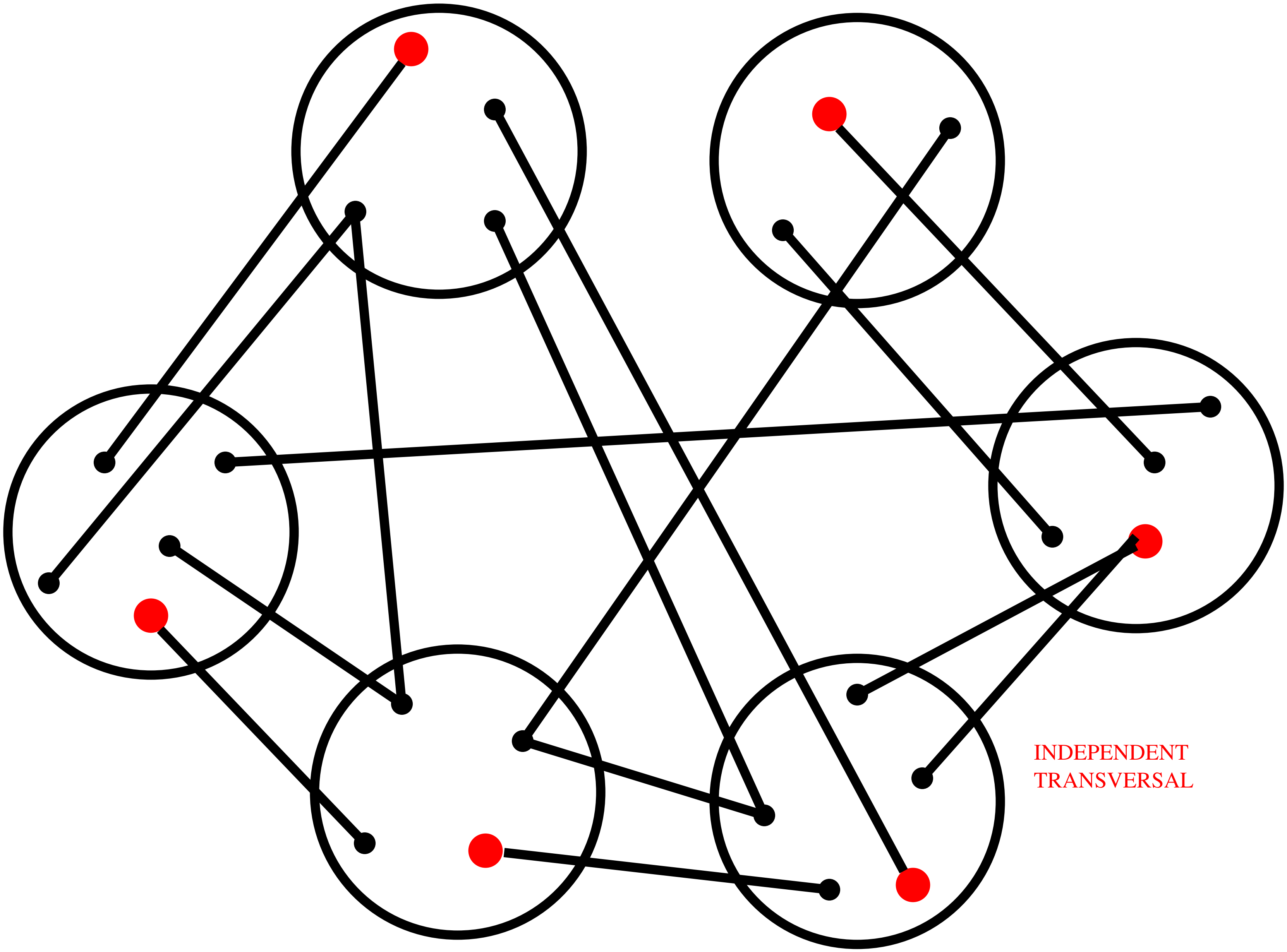
University of Waterloo






should NOT be on a
committee together!




INDEPENDENT
TRANSVERSAL

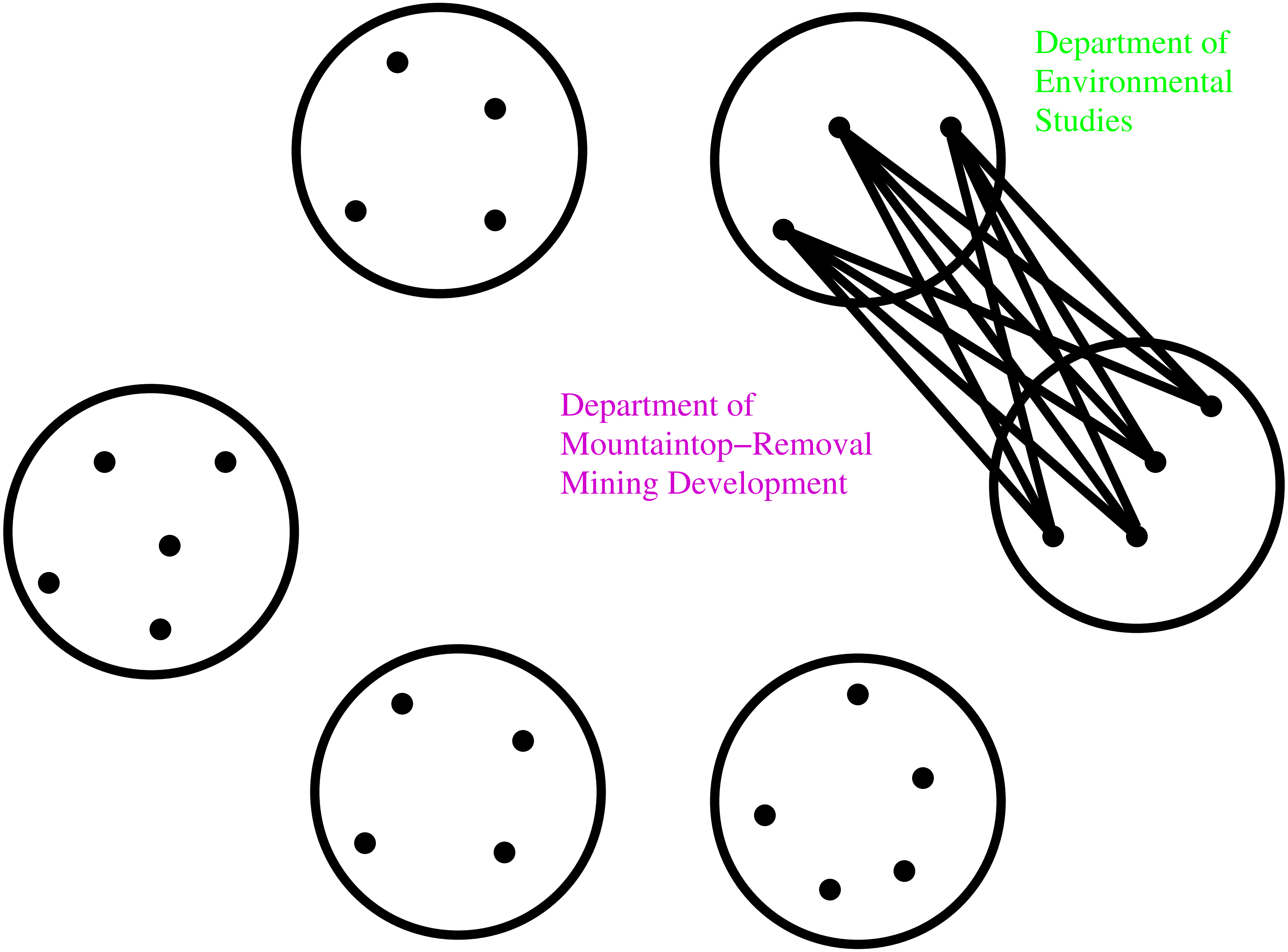
Independent transversals

An independent transversal in a vertex-partitioned graph G is a subset T of vertices such that

- no edge of G joins two vertices of T (independent)
- T contains exactly one vertex from each partition class (transversal)

IN OTHER WORDS: a good committee.

When does a good committee exist?


Department of
Environmental
Studies
Department of
Mountaintop–Removal
Mining Development

When does a good committee exist?

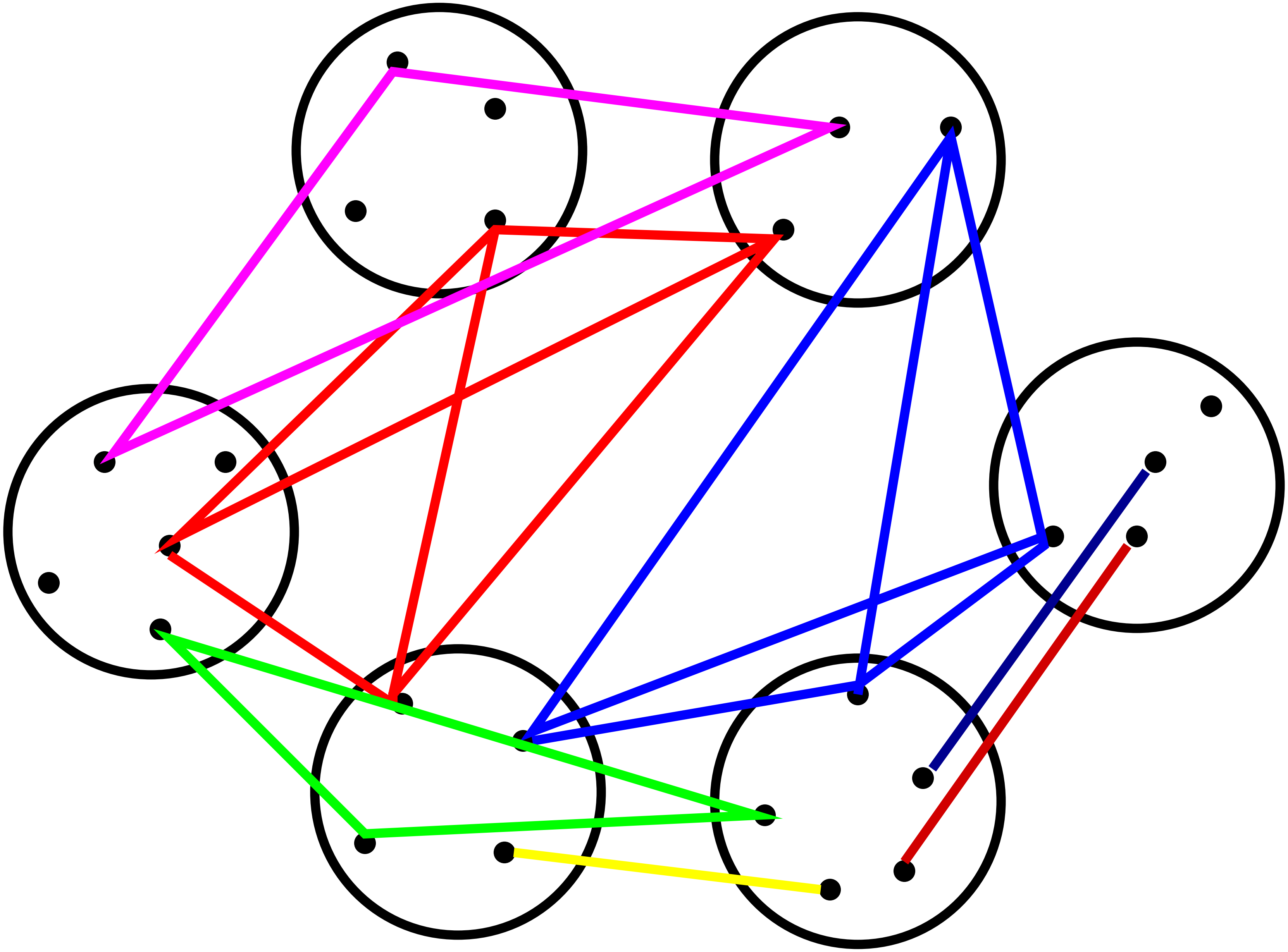
- Not always.

The Unhappy Families Case

Suppose that

- Each faculty member belongs to one of a number of (unhappy) FAMILIES.
- Two faculty members from the same family cannot agree on any matter.

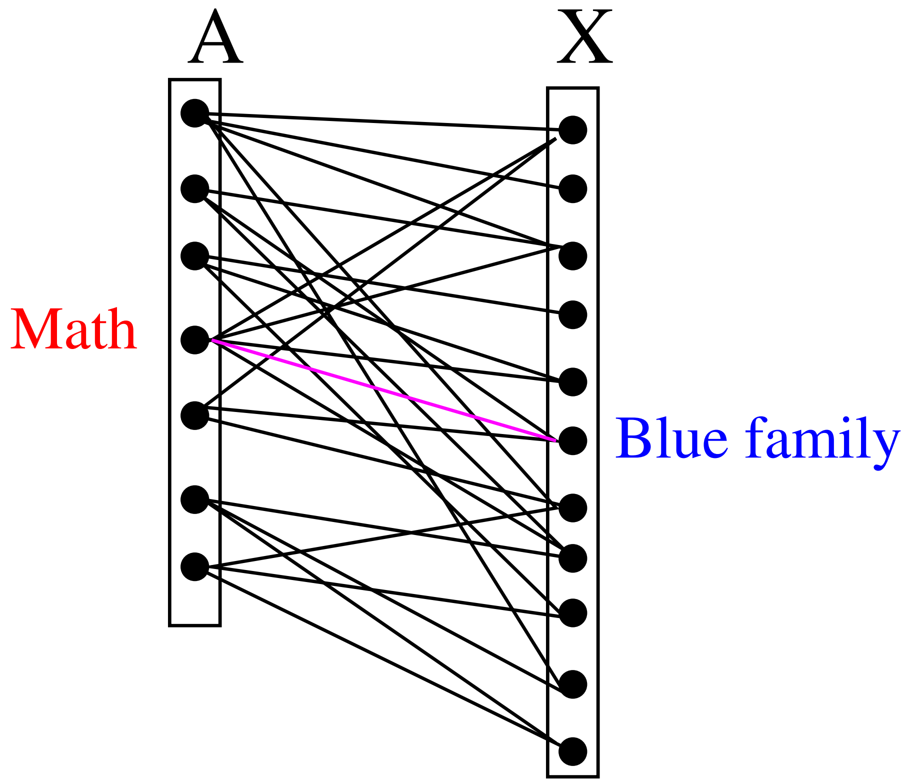
(We may assume no two members of the same family are in the same department.)



We can model this case as a bipartite graph B with vertex classes

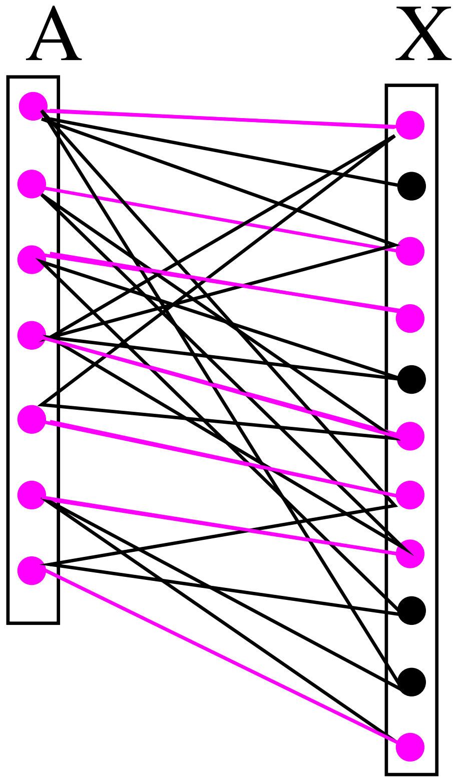
- A: the set of departments
- X: the set of families

where each faculty member y is represented by an edge joining the department containing y to the family containing y.



A good committee corresponds to a set of disjoint edges in B that covers A.

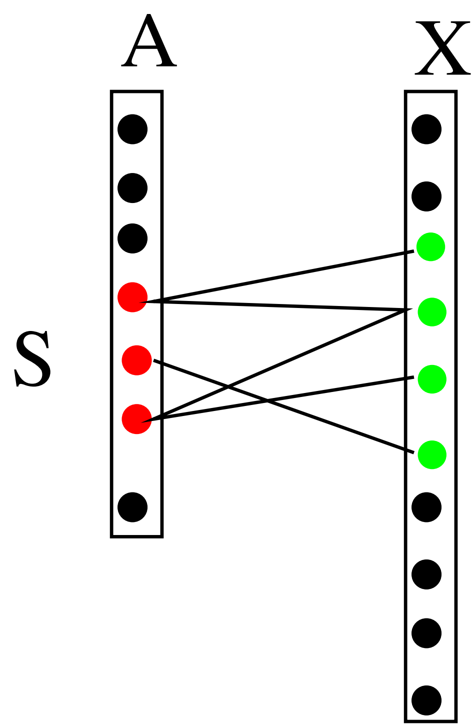
IN OTHER WORDS: a good committee corresponds to a complete matching from A to X in B.

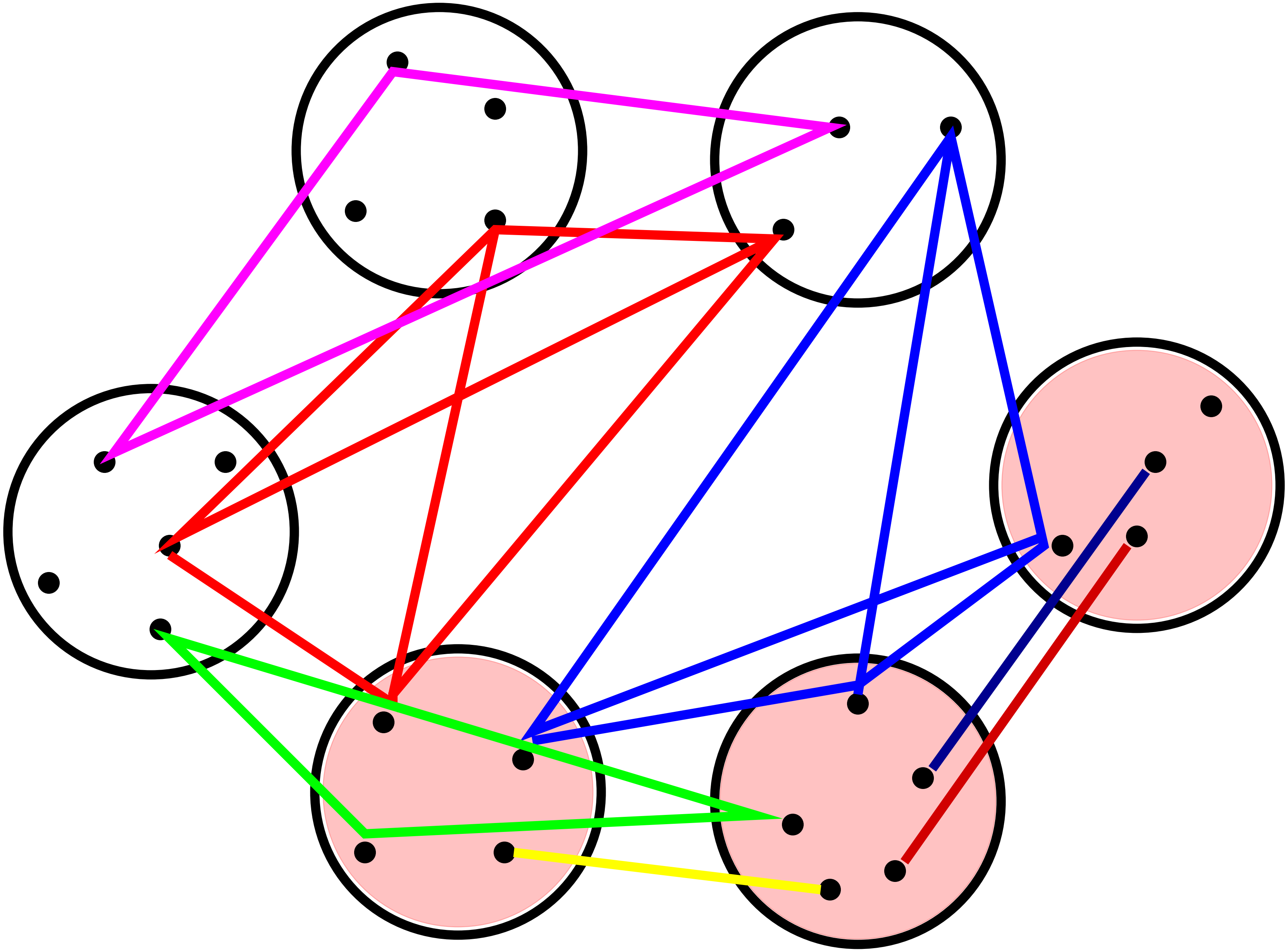


Hall's Theorem

THEOREM: The bipartite graph B has a complete matching if and only if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ satisfies

$$|\Gamma(S)| \geq |S|.$$





When does a good committee exist?

- Not always.

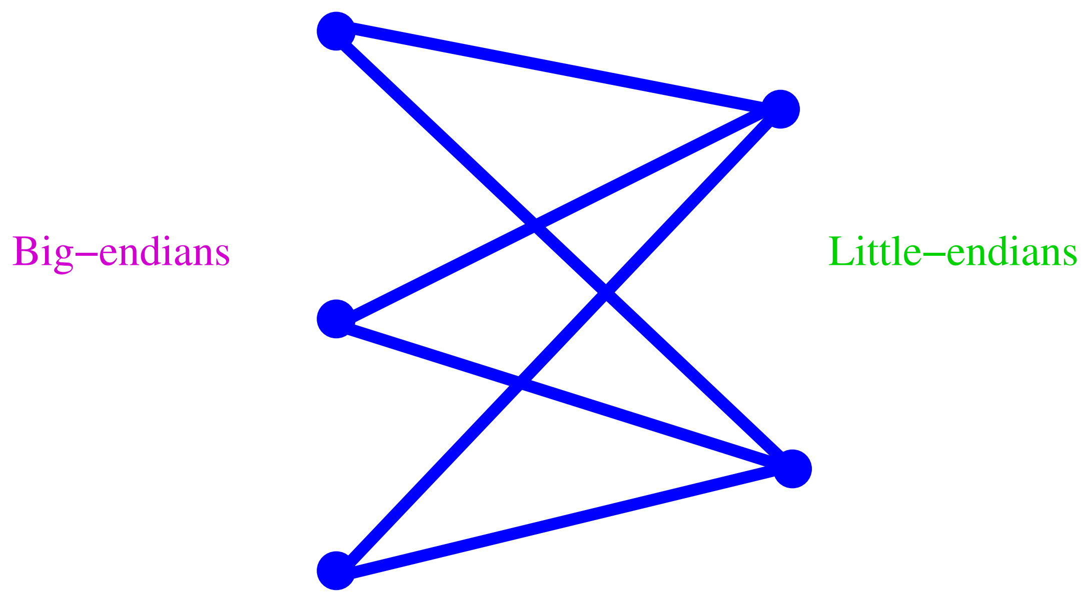
- In the Unhappy Families case: when every subset S of departments contains representatives from at least $|S|$ families. (Hall's Theorem. Moreover a good committee can be found efficiently if it exists.)

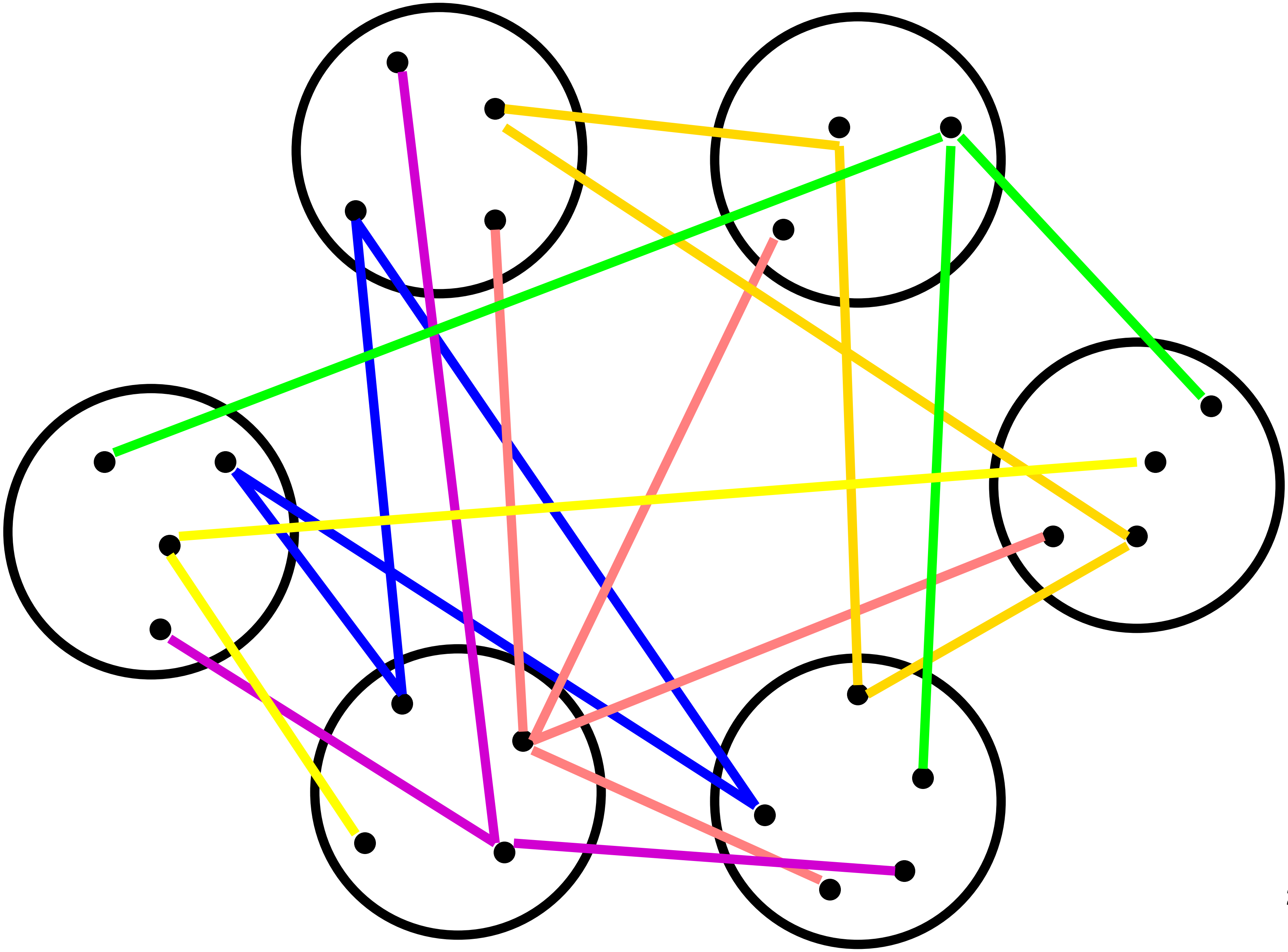
The Big Issues Case

Suppose that

- Each faculty member has a deeply held opinion about one particular (two-sided) ISSUE.
- Each issue captivates at most one faculty member per department.
- Two faculty members having opposite views on the same issue cannot agree on any other matter either.

A Typical Issue





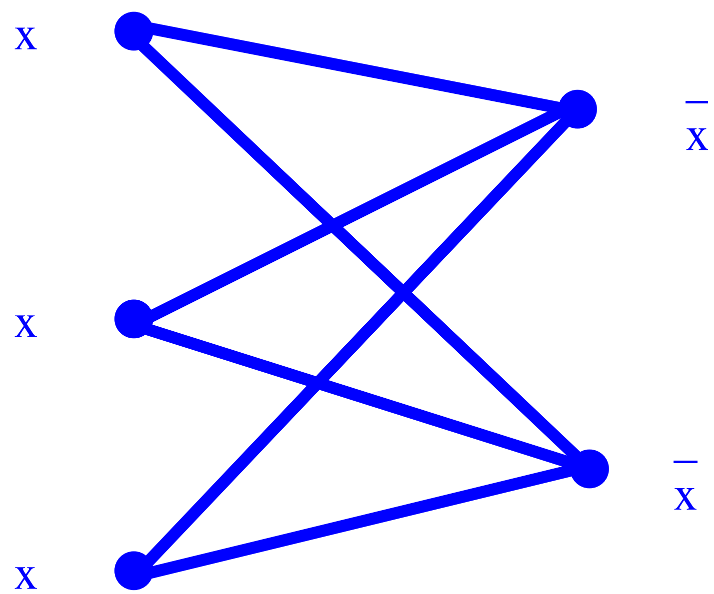
The SAT problem

Given a Boolean formula, does it have a satisfying truth assignment?

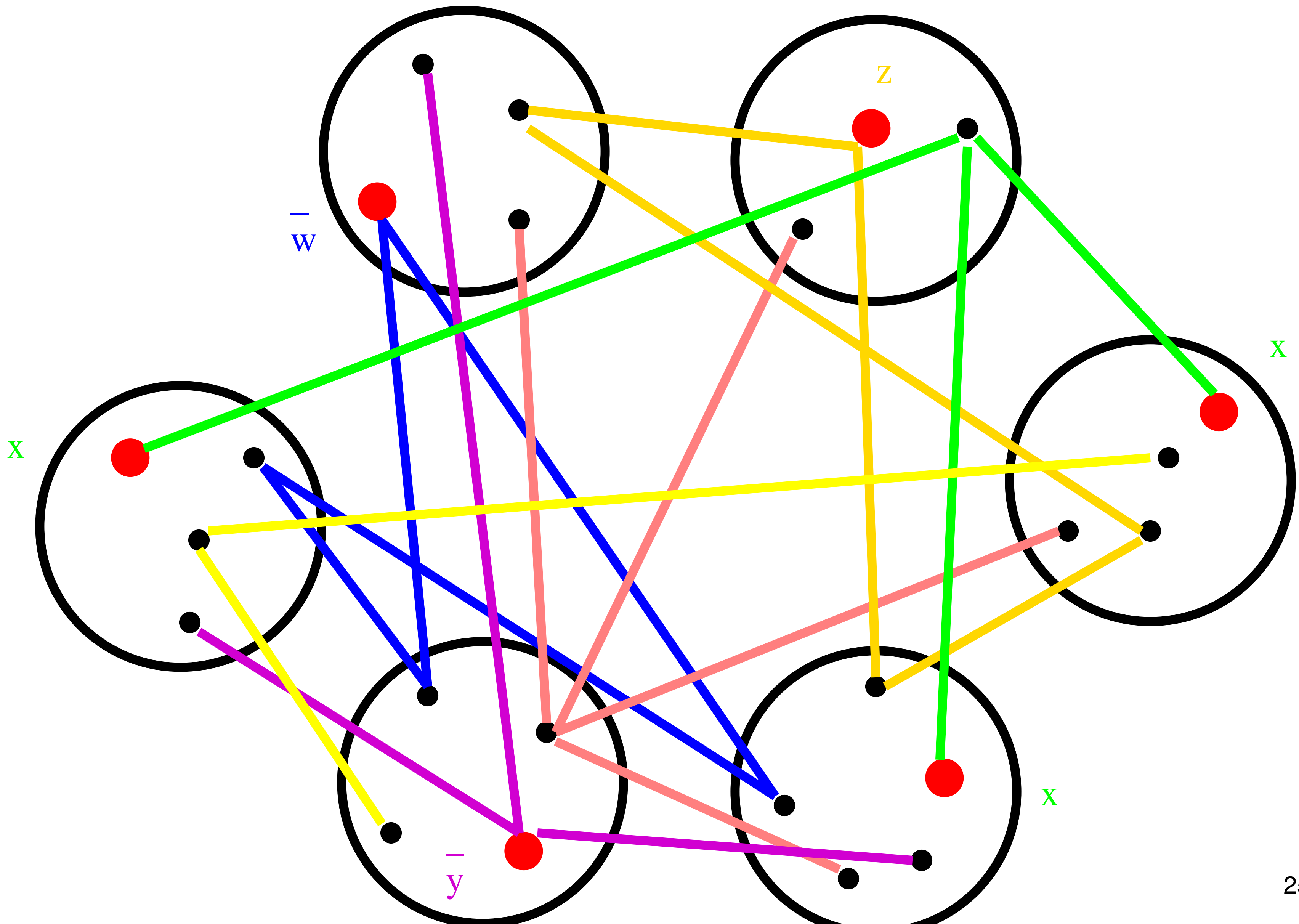
$$(x_1 \vee \bar{x_4} \vee x_7) \wedge (\bar{x_1} \vee \bar{x_3} \vee x_2) \wedge (x_3 \vee \bar{x_2}) \wedge (x_5 \vee x_6 \vee \bar{x_2})$$

- Clauses correspond to partition classes (departments)
- Variables correspond to issues

A Typical Variable



A satisfying truth assignment corresponds to an independent transversal.



z
$\bar{w}$
x
x
x
$\bar{y}$

When does a good committee exist?

- Not always.
- In the Unhappy Families case: when every subset S of departments contains representatives from at least $|S|$ families. (Hall's Theorem. Moreover a good committee can be found efficiently if it exists.)
- the Big Issues case: same as deciding the SAT problem. (So we cannot expect an efficient characterization.)

We will see

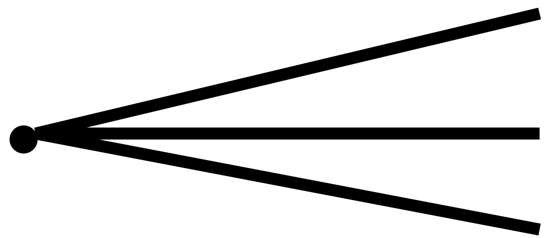
- some sufficient conditions that guarantee the existence of an independent transversal in a given vertex-partitioned graph

- some ideas of the proofs of these results

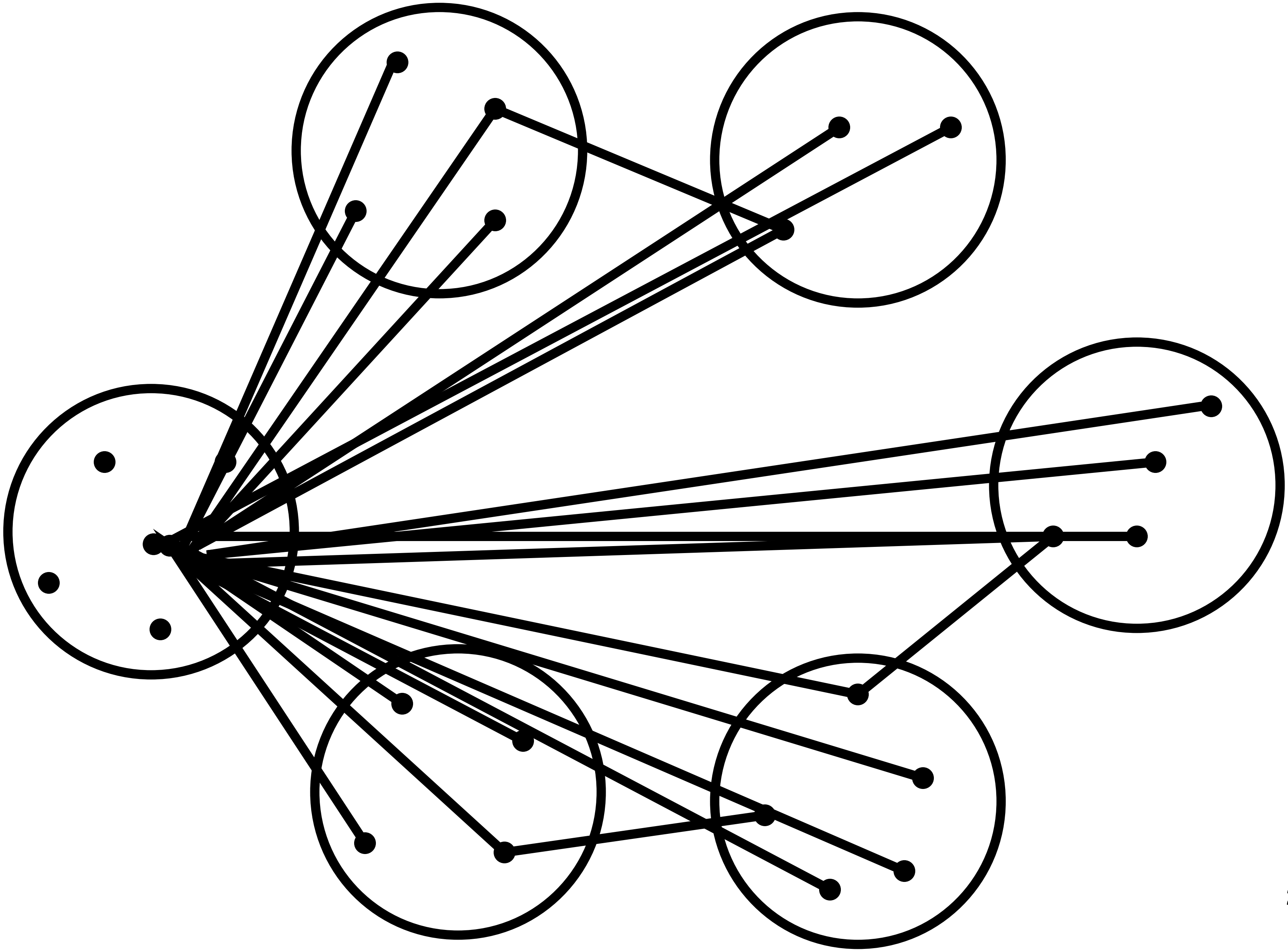
- some applications.

Maximum Degree

Suppose every vertex has degree at most d.



“limited personal conflict”: no faculty member is in conflict with more than d others.



QUESTION: When the graph has maximum degree at most d, how big do the partition classes need to be in terms of d to guarantee the existence of an independent transversal?

This question was first introduced and studied by Bollobás, Erdös and Szemerédi (1975). Also

- Jin (1992)
- Yuster (1997)
- Alon (2002)
- Szabó and Tardos (2003): gave an example with
  - maximum degree d
  - $2d$ classes
  - each class of size $2d - 1$

  having NO independent transversal.

THEOREM: Partition classes of size $2d$ suffice.

When does a good committee exist?

- Not always.
- In the Unhappy Families case: when every subset S of departments contains representatives from at least $|S|$ families. (Hall's Theorem. Moreover a good committee can be found efficiently if it exists.)
- the Big Issues case: same as deciding the SAT problem. (So we cannot expect an efficient characterization.)
- if no faculty member conflicts with more than d others, and departments all have size at least $2d$.

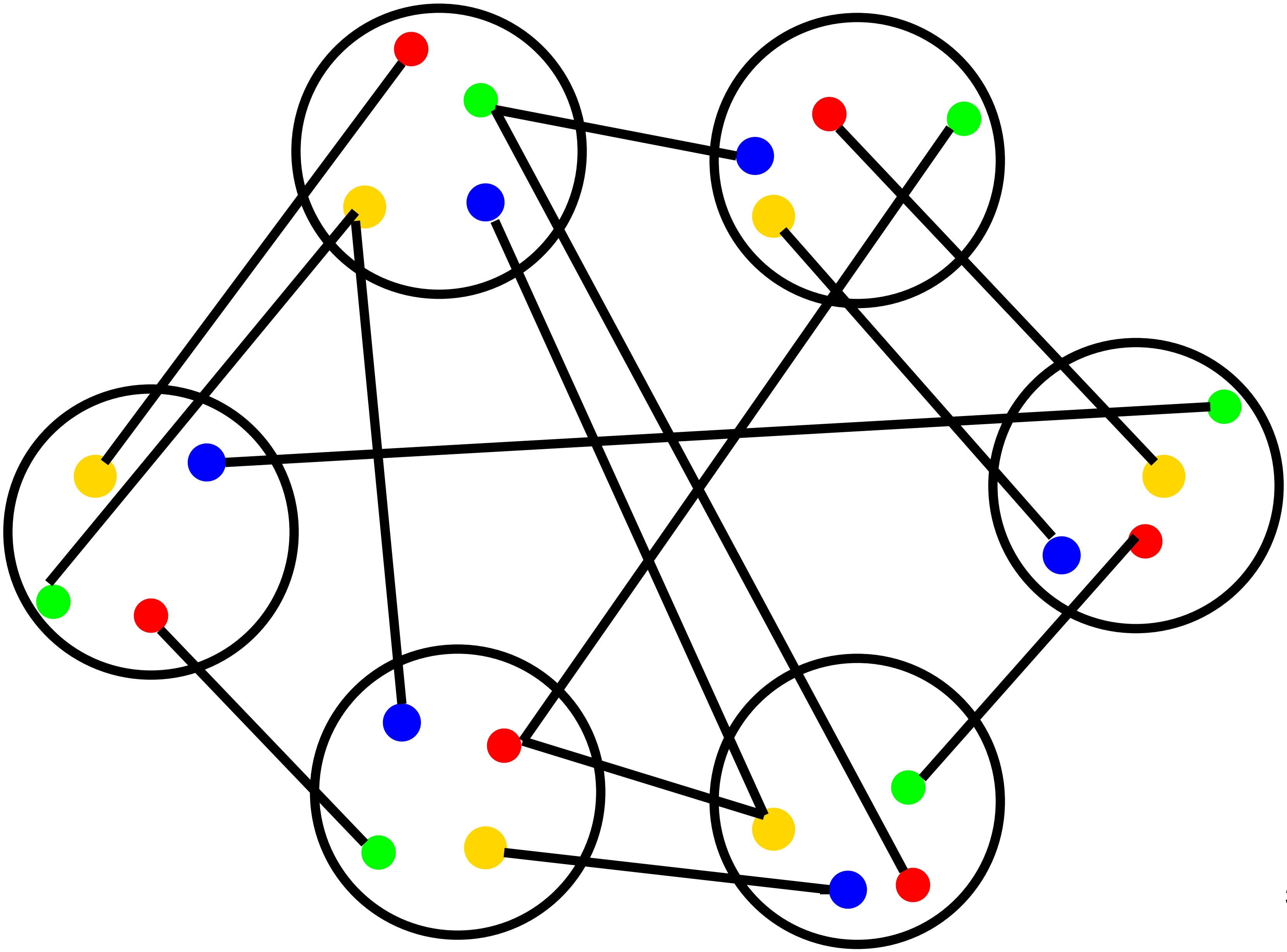
The Happy Dean Problem

Suppose every vertex has degree at most d.

QUESTION: What conditions will guarantee the existence of a PARTITION into independent transversals?

Some obvious necessary conditions:

- all partition classes have the same size
- partition classes have size at least $2d$.



Strong Colouring

Let G be a graph with n vertices, where $r|n$. We say G is *strongly r-colourable* if for every vertex partition of G into classes of size r, there exist r DISJOINT independent transversals.

If $r \not| n$ then G is strongly r-colourable if by adding isolated vertices until $r|n'$ we obtain a strongly r-colourable graph.

The *strong chromatic number* $s\chi(G)$ of G is the smallest r for which G is strongly r-colourable.

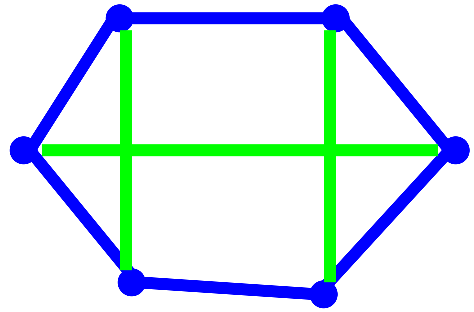
QUESTION: How does the strong chromatic number depend on the maximum degree?

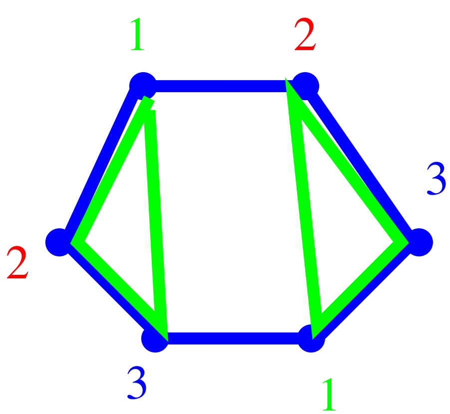
Strong chromatic number was first introduced and studied by Alon (1988) and Fellows (1990).

In 1992, Alon proved a linear upper bound for the strong chromatic number in terms of the maximum degree d for any graph:

$$s\chi(G) \leq cd.$$

QUESTION: What is the correct value of c?

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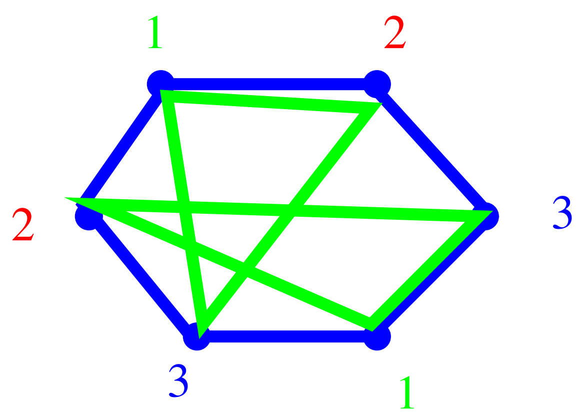
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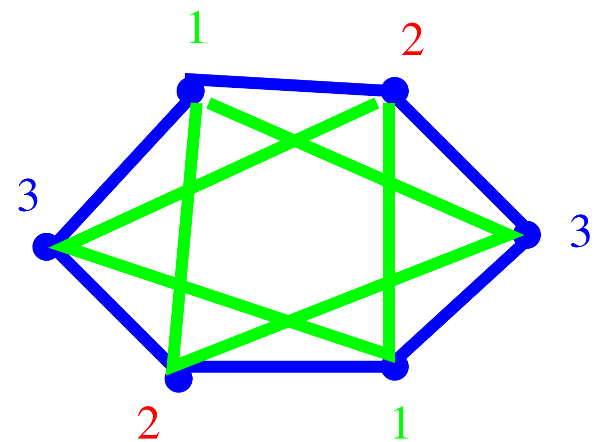
1
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1

THEOREM: Every graph with maximum degree d satisfies

$$s\chi(G) \leq 3d - 1.$$

THEOREM: Every graph with maximum degree d satisfies

$$s\chi(G) \leq (\alpha + o(1))d,$$

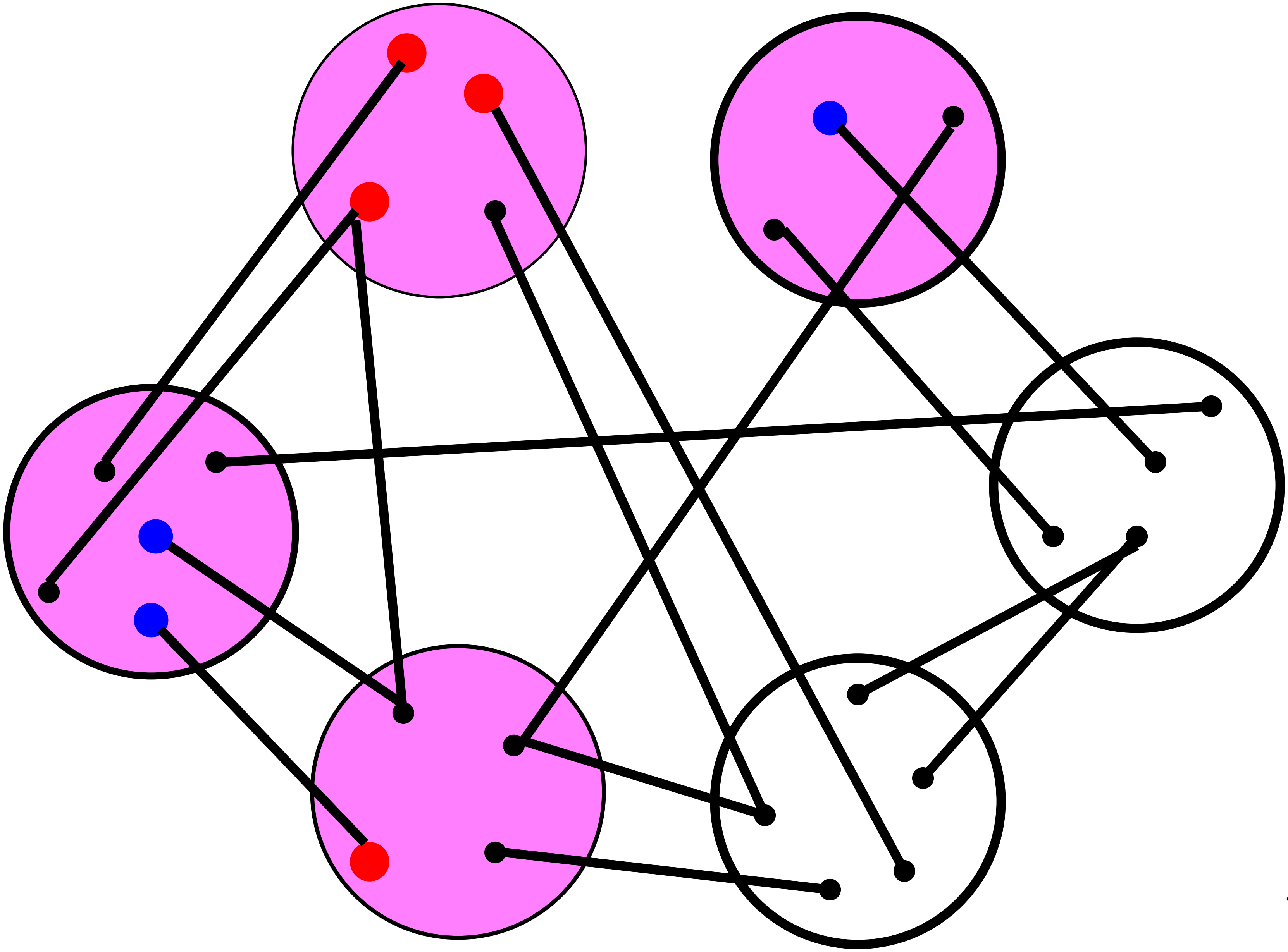
where $\alpha = 2.73\ldots$

Another Sufficient Condition

THEOREM (Aharoni, PH): Let G be a graph with vertex classes $V_1, \ldots, V_m$. Suppose that for every $I \subset \{1, \ldots, m\}$ there exists an independent set S_I in $G_I = G[\cup_{i \in I} V_i]$ such that

every independent set T in G_I of size at most $|I| - 1$ can be extended by a vertex of S_I.

Then G has an independent transversal.



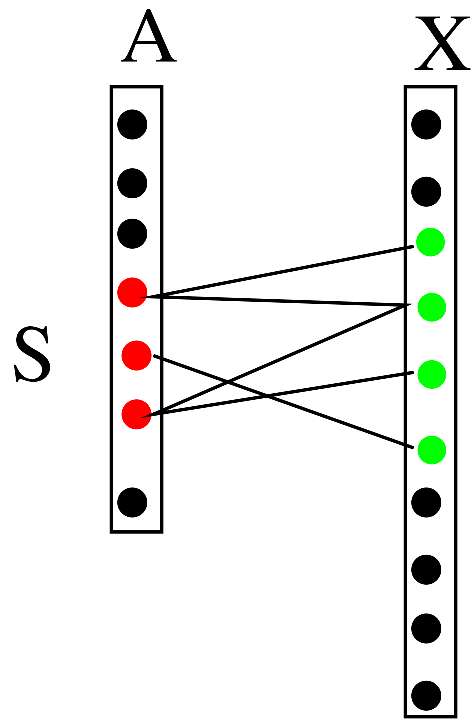
An Application

We can use this theorem to obtain a generalisation of Hall's Theorem for matchings in bipartite graphs to hypermatchings in bipartite hypergraphs.

Hall's Theorem

THEOREM: The bipartite graph G has a complete matching if and only if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ is big enough.

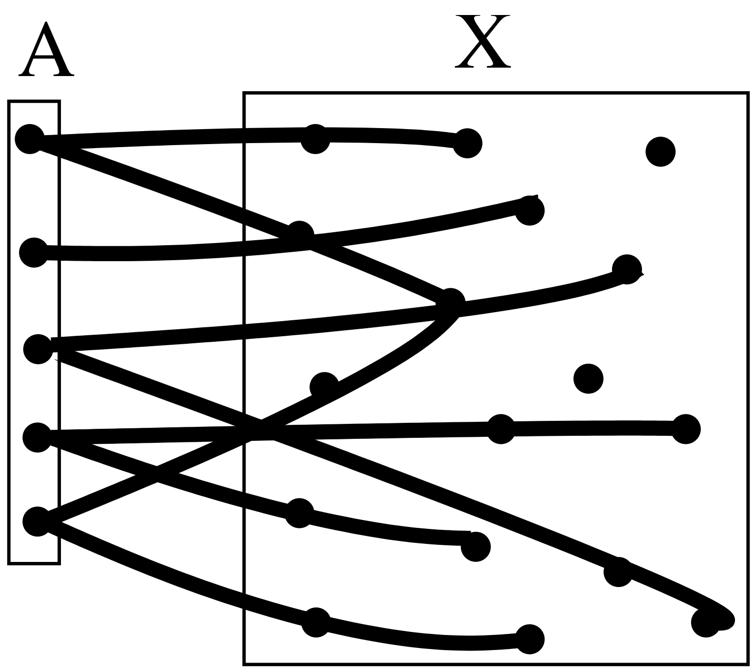
Here big enough means $|\Gamma(S)| \geq |S|$.



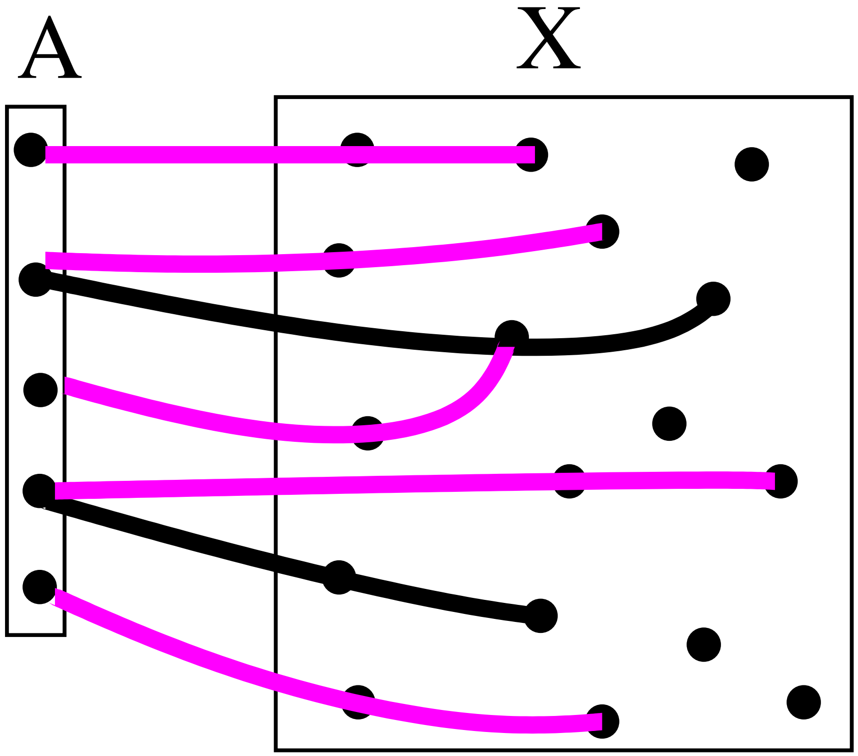
A generalisation to hypergraphs

def: A 3-uniform hypergraph consists of a set V of vertices and a set H of hyperedges, where each hyperedge is a subset of V of size three.

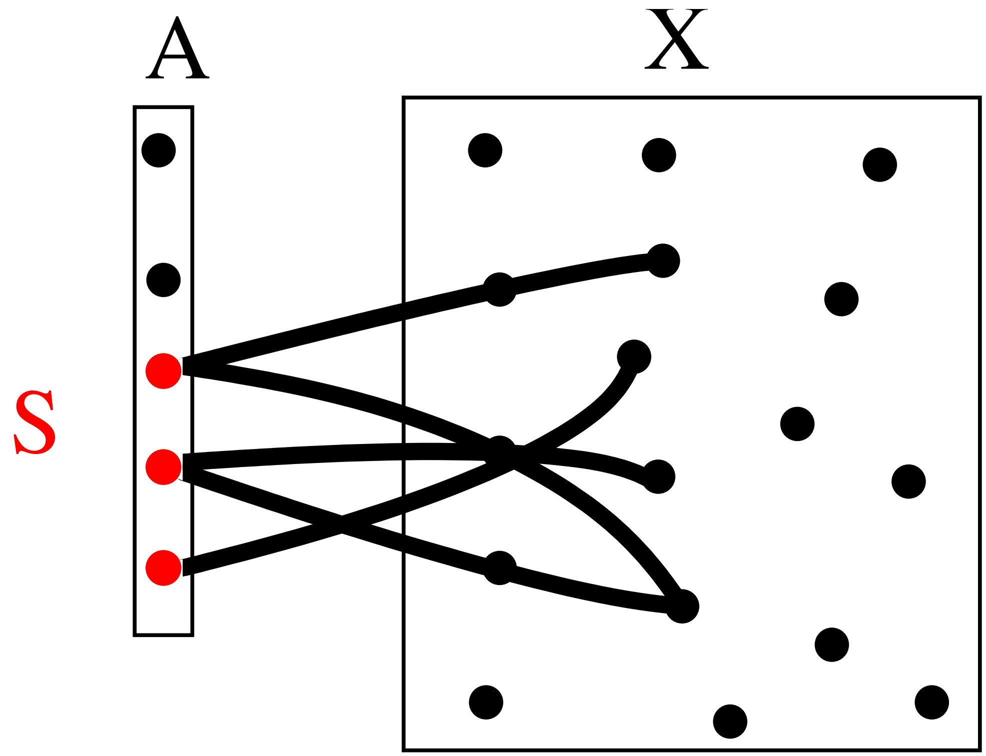
def: A bipartite 3-uniform hypergraph:

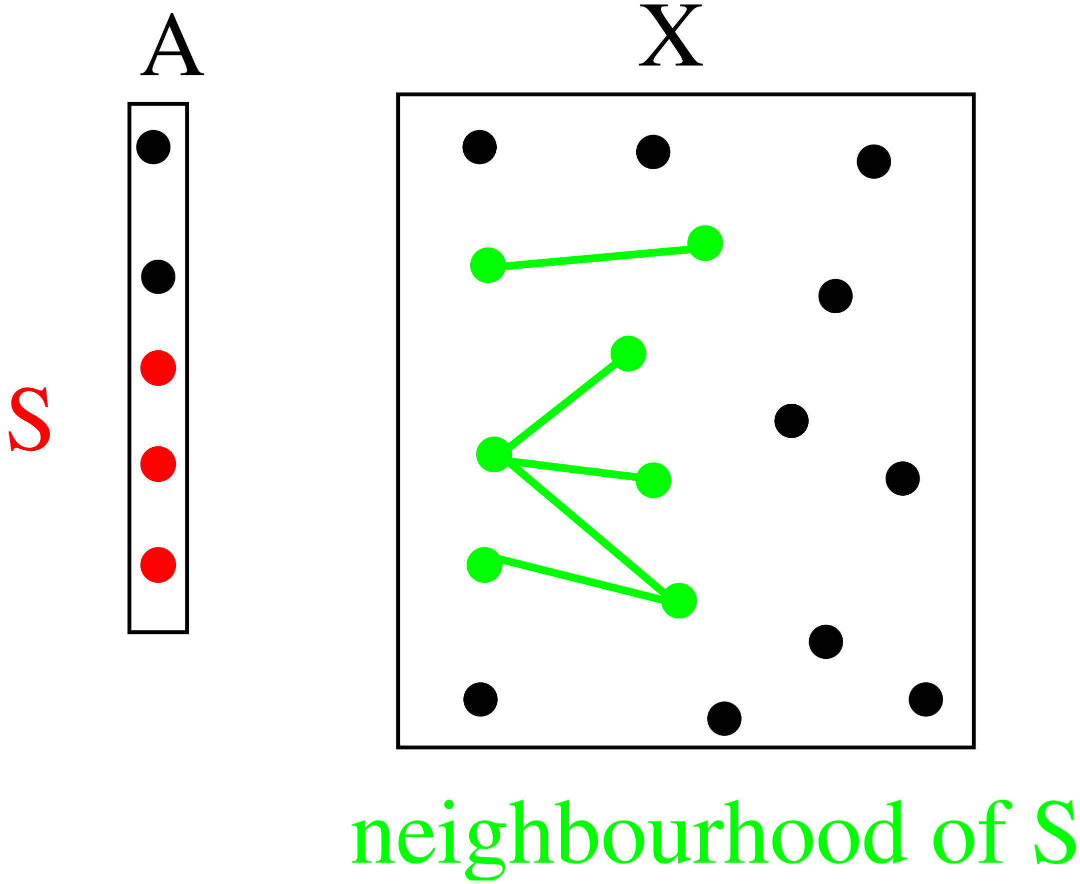


def: A complete hypermatching:

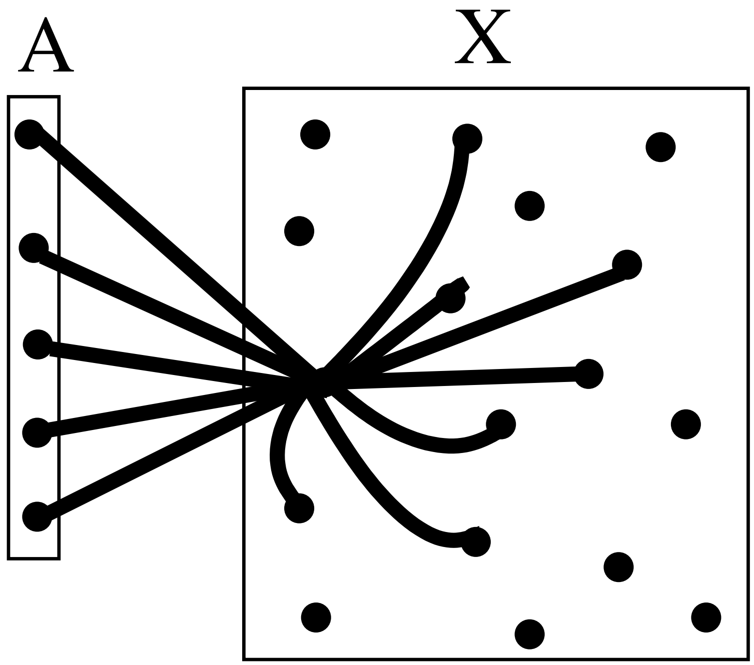


def: The neighbourhood of the subset S of A is the graph with vertex set X and edge set $\{\{x,y\} : \{z,x,y\} \in H$ for some $z \in S\}$.

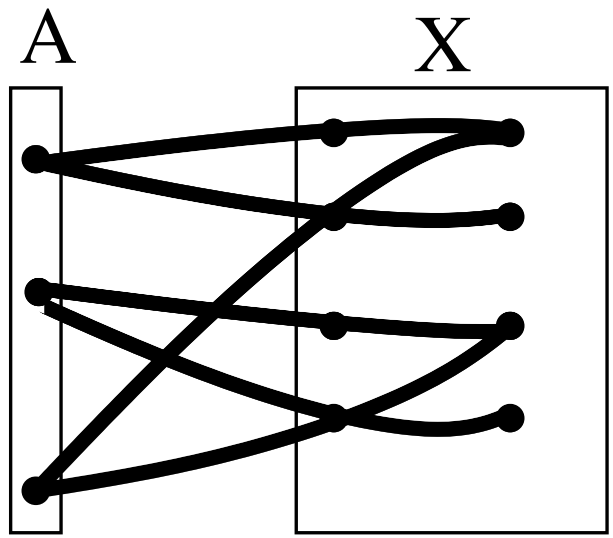



A
X
S
neighbourhood of S

What should big enough mean?



Big enough $=$ Has a large matching

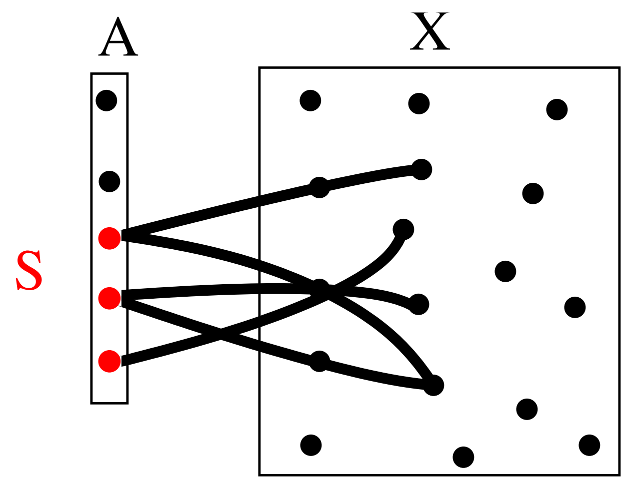


$\Gamma(S)$ has a matching of size at least $2(|S|-1)$ for each S, but there is **NO** complete hypermatching.

Hall's Theorem for 3-uniform hypergraphs

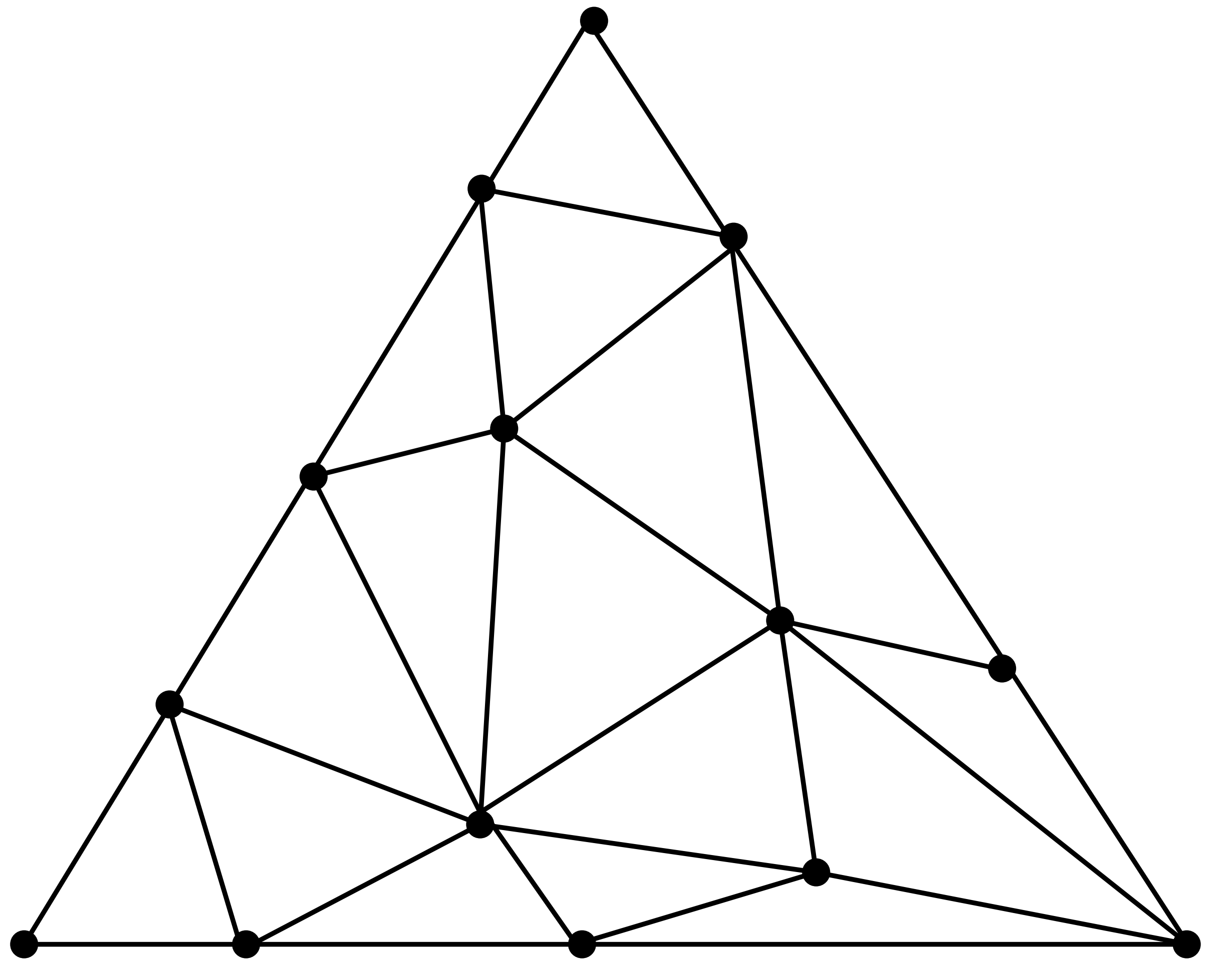
THEOREM: The bipartite 3-uniform hypergraph G has a complete hypermatching if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ is big enough.

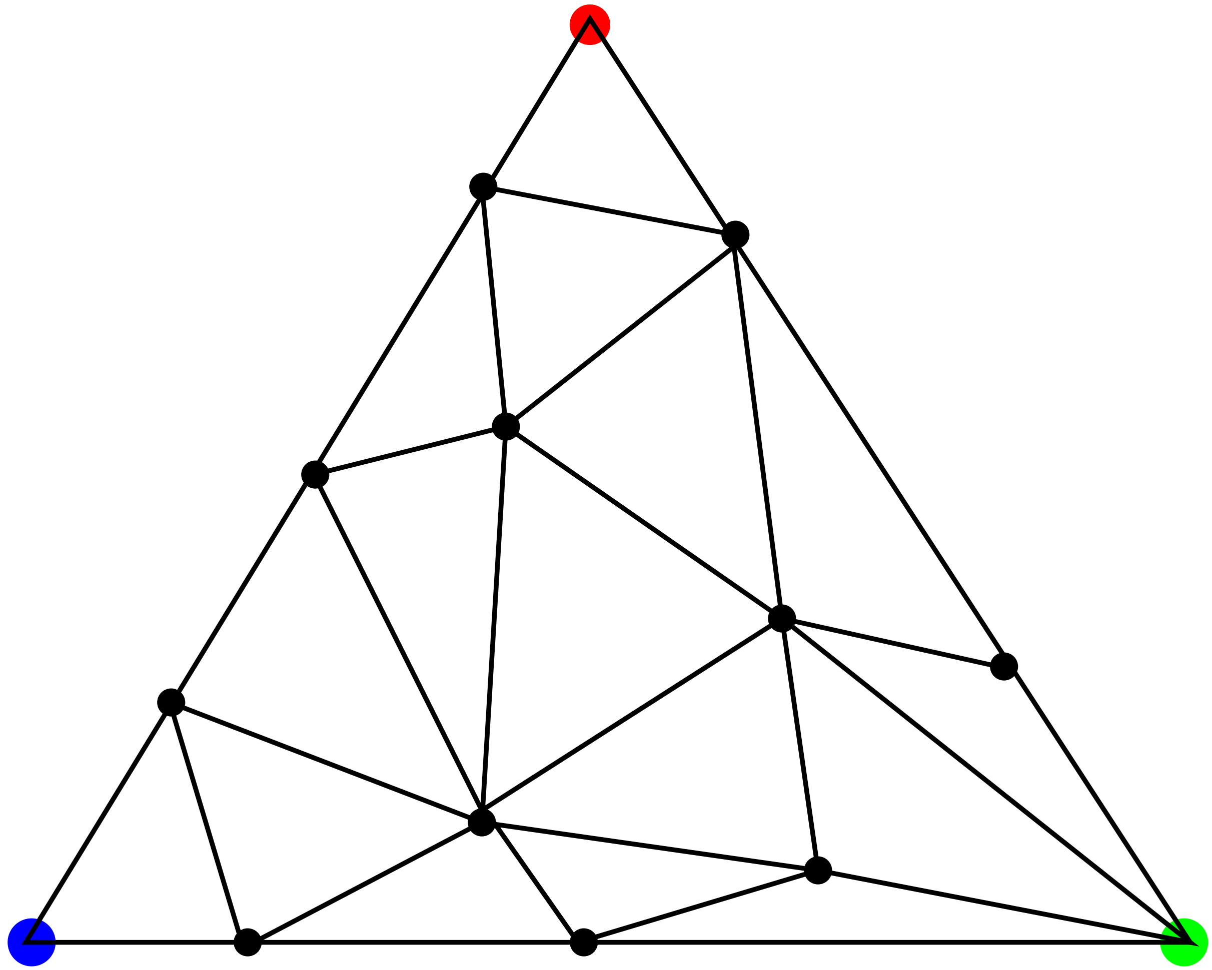
Here big enough means has a matching of size at least $2(|S|-1)+1$.

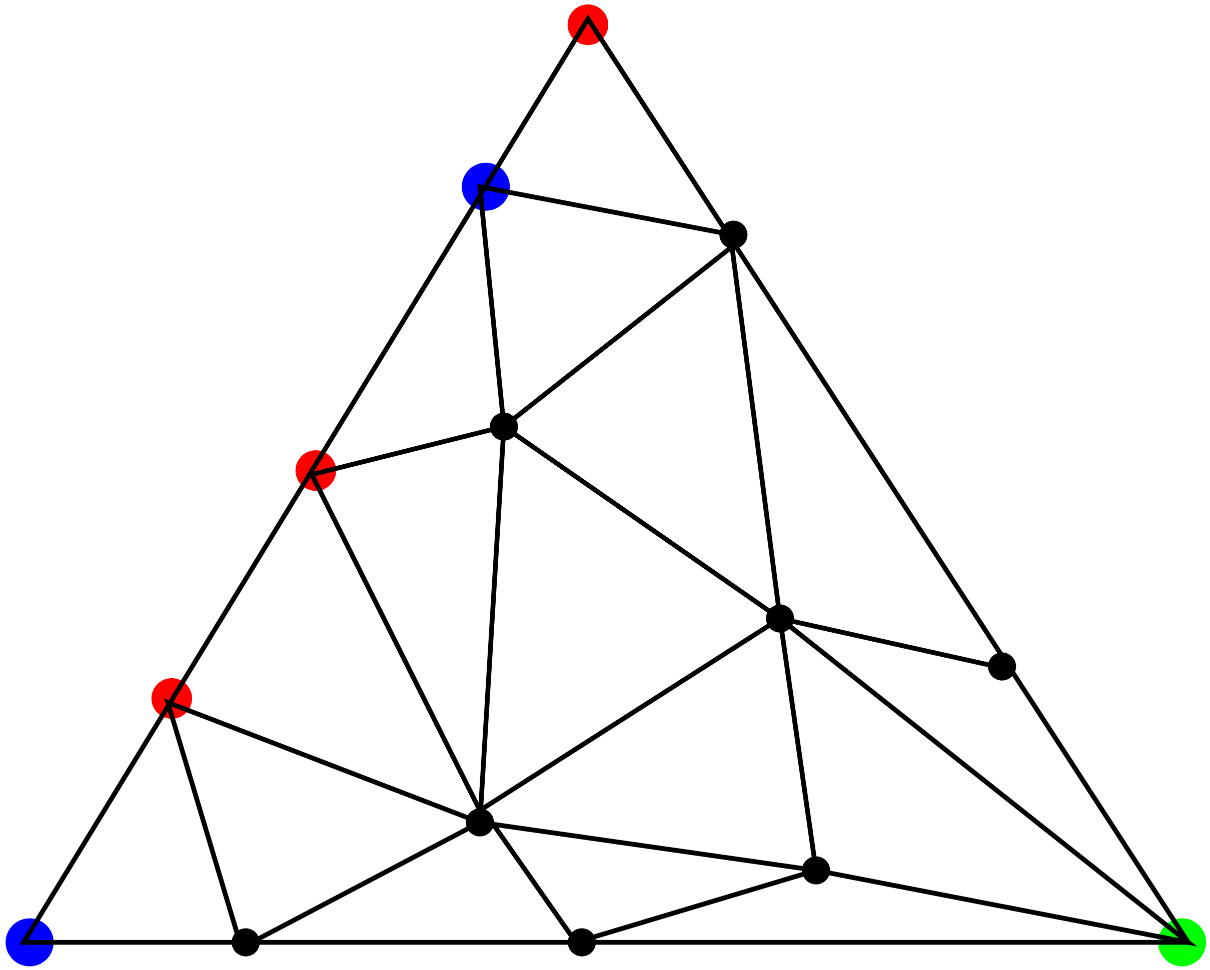


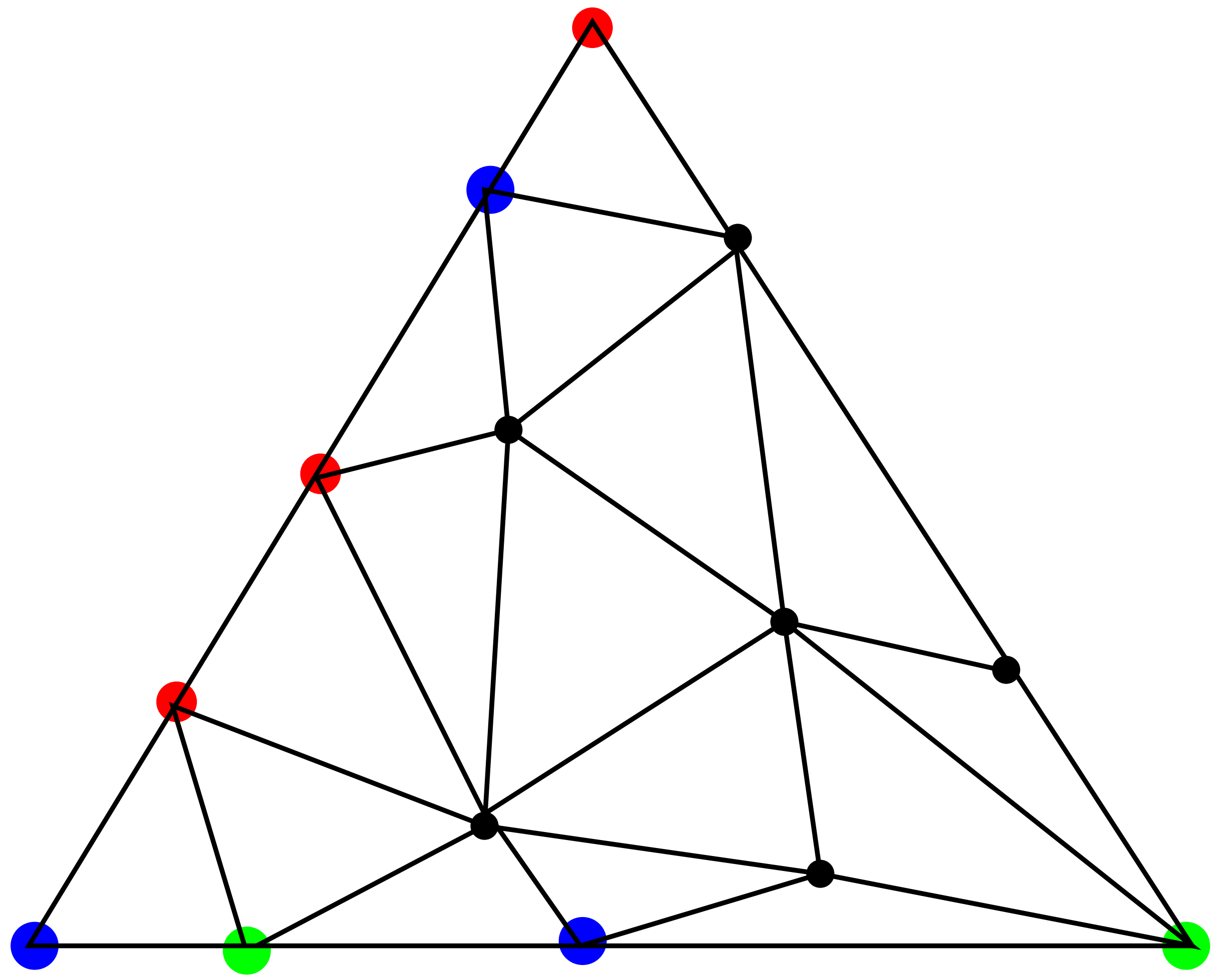
Proof

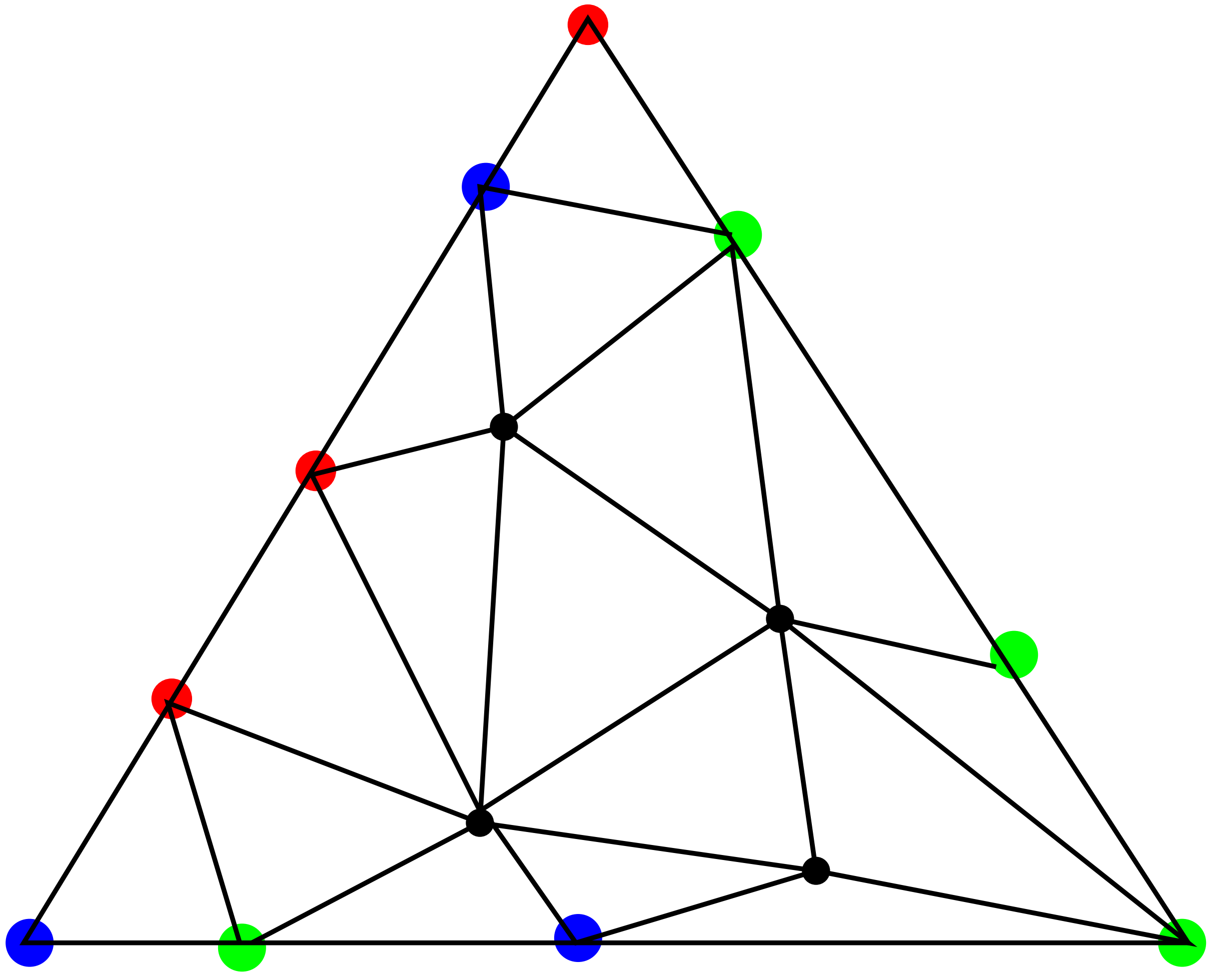
- We'll see the idea of the proof of the theorem, specialised to our particular application of Hall's Theorem for hypergraphs.
- The proof uses Sperner's Lemma.

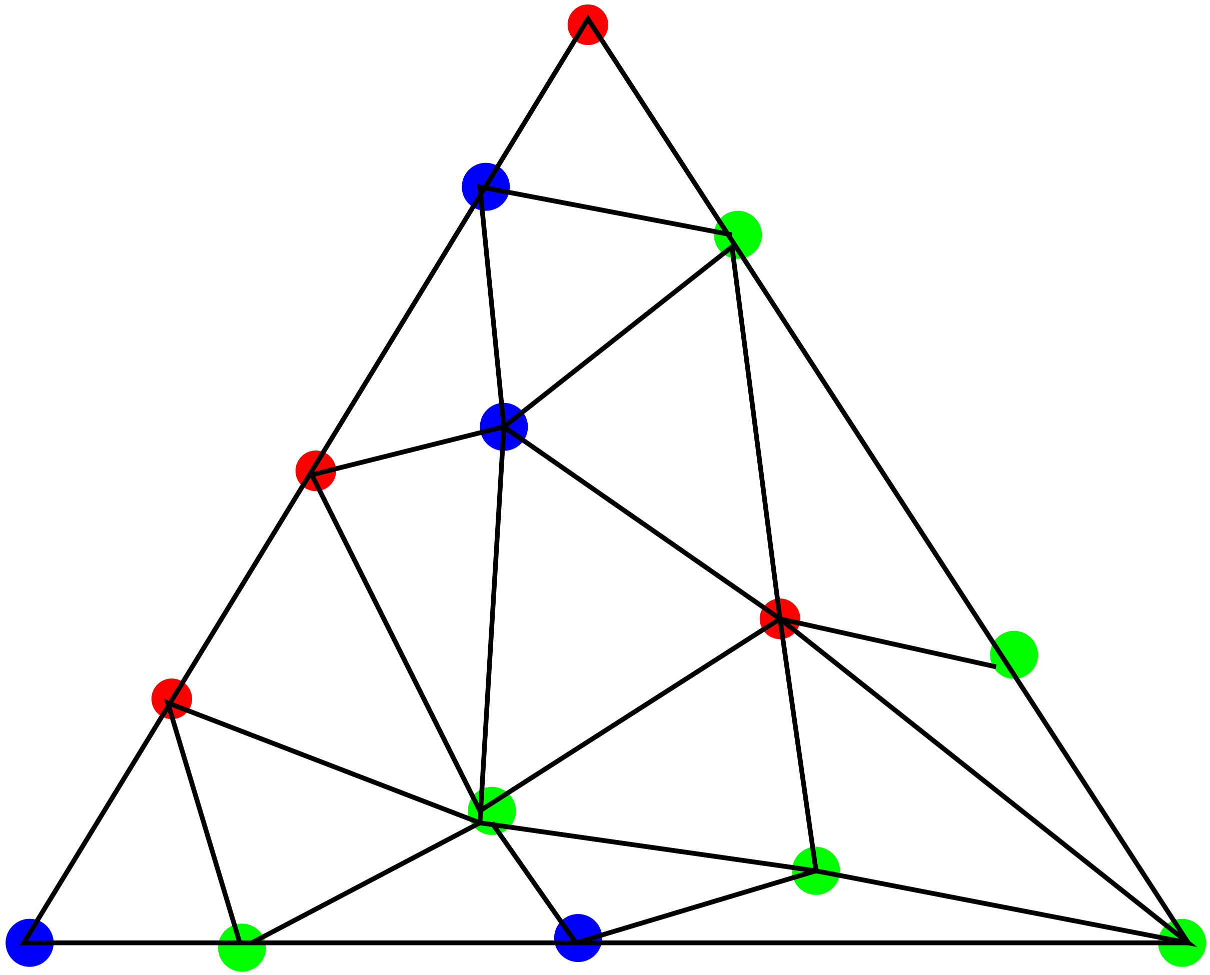


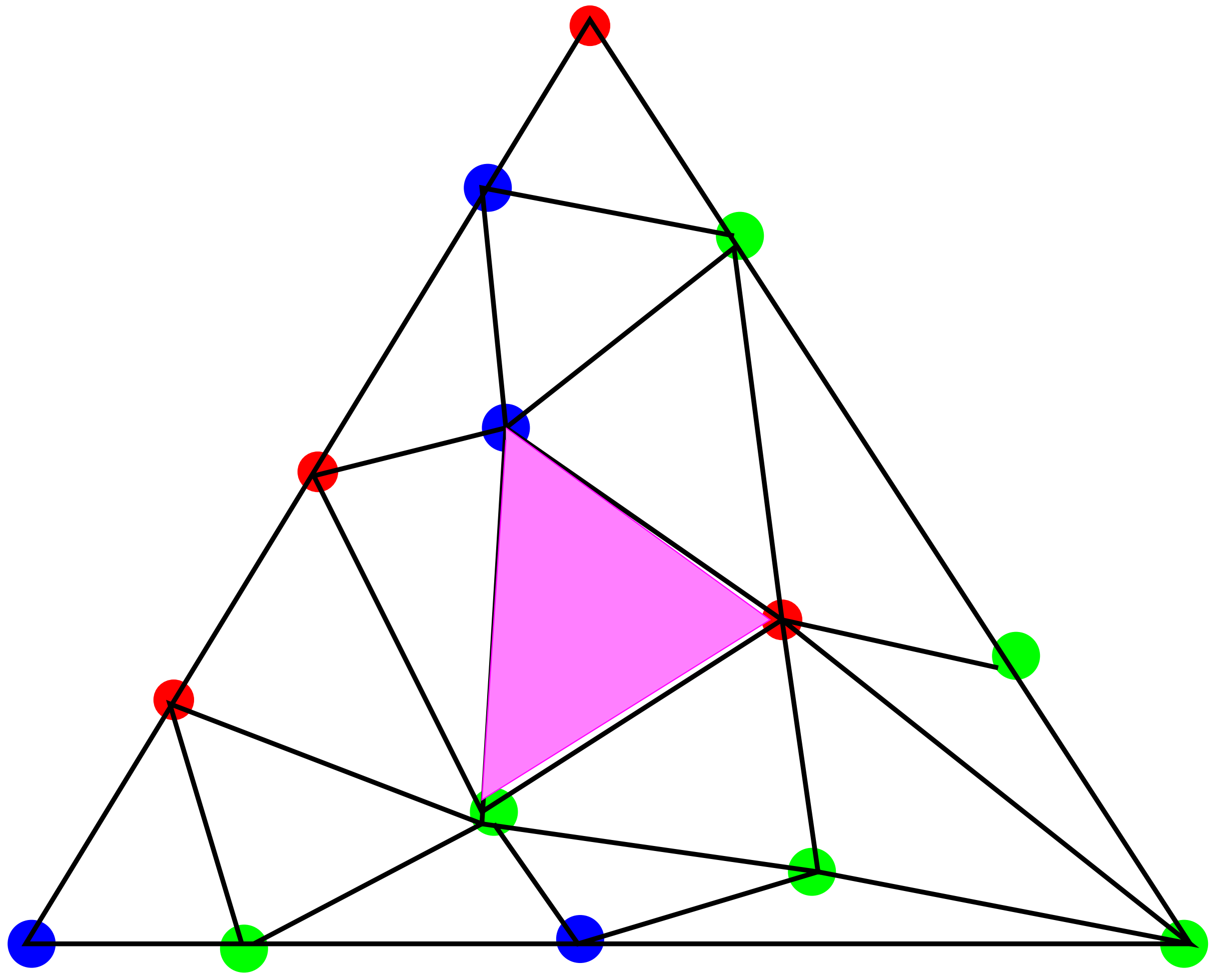






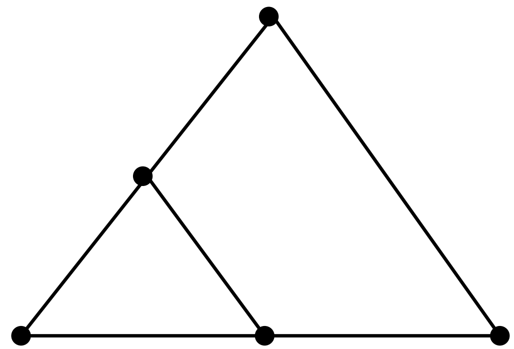




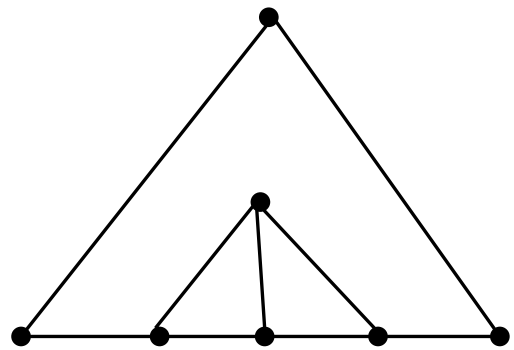


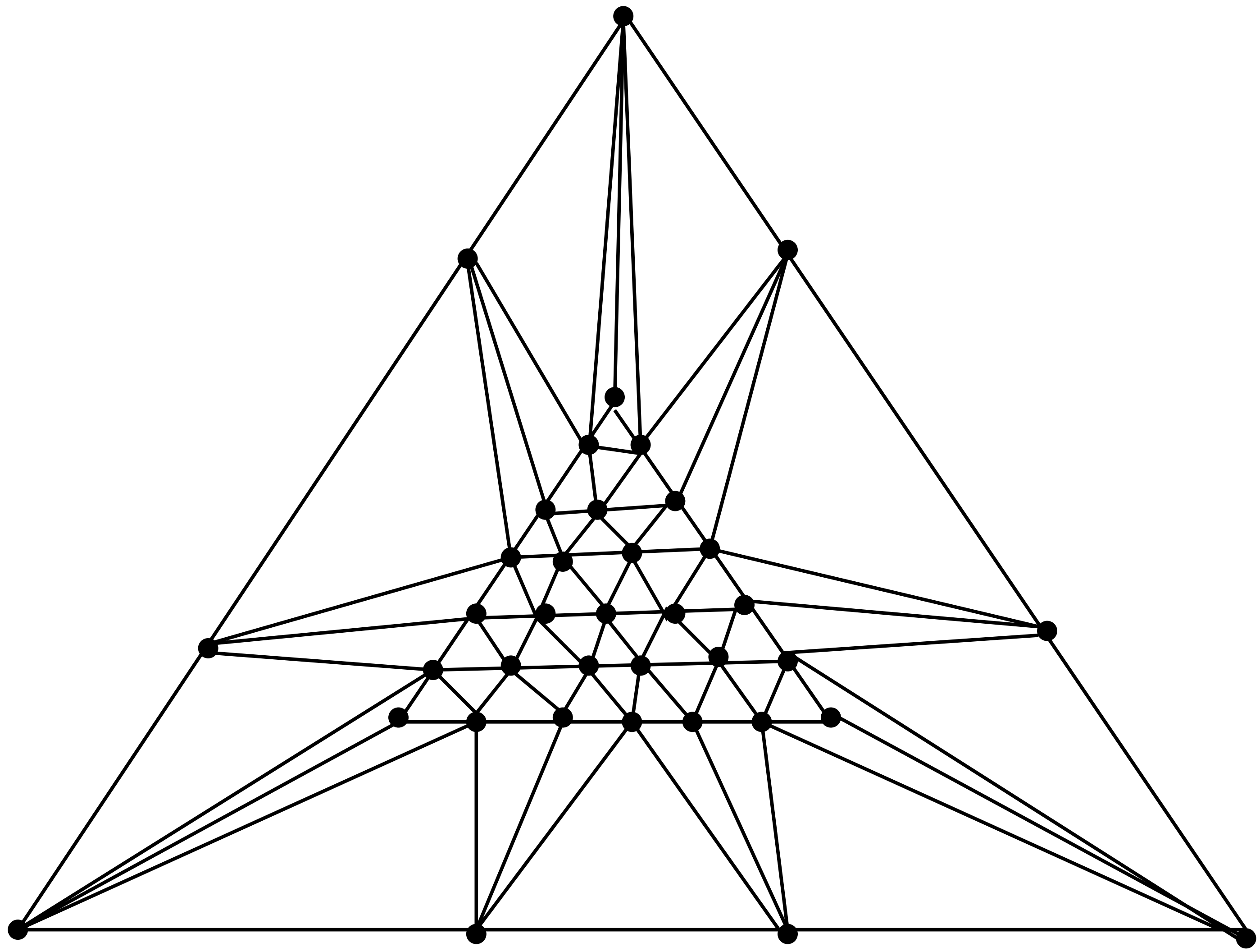
A Special Triangulation

NO



NO





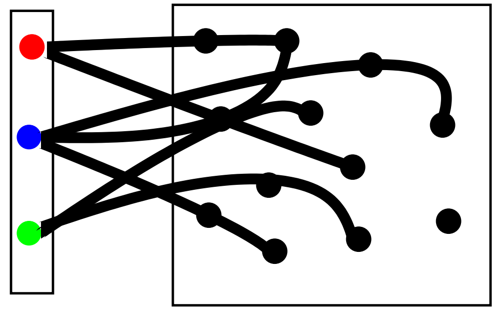
Idea of Proof

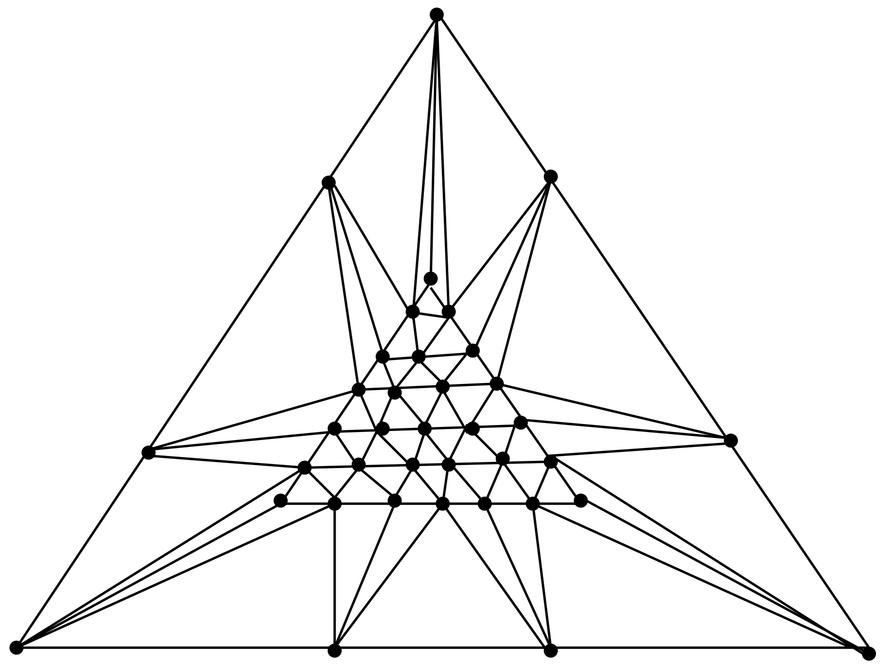
Let H be a bipartite hypergraph with vertex classes A and X. Let T be the special triangulation of the n-simplex, where $n = |A| - 1$.

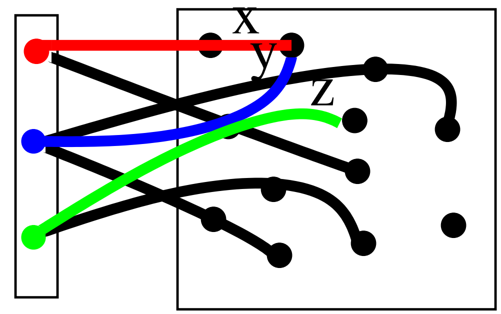
We will label the points of T with edges of $\Gamma(A)$, and colour each of them with the corresponding vertex of A, such that

- edges labelling adjacent points in T are disjoint,
- the resulting colouring is a Sperner colouring.

Then the multicoloured simplex given by Sperner's Lemma is a complete hypermatching in H.



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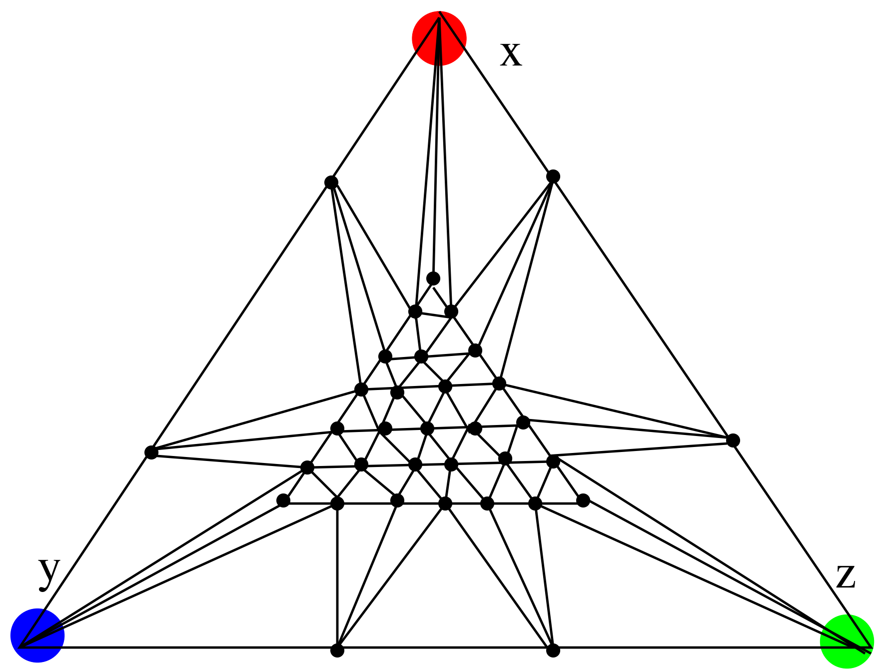
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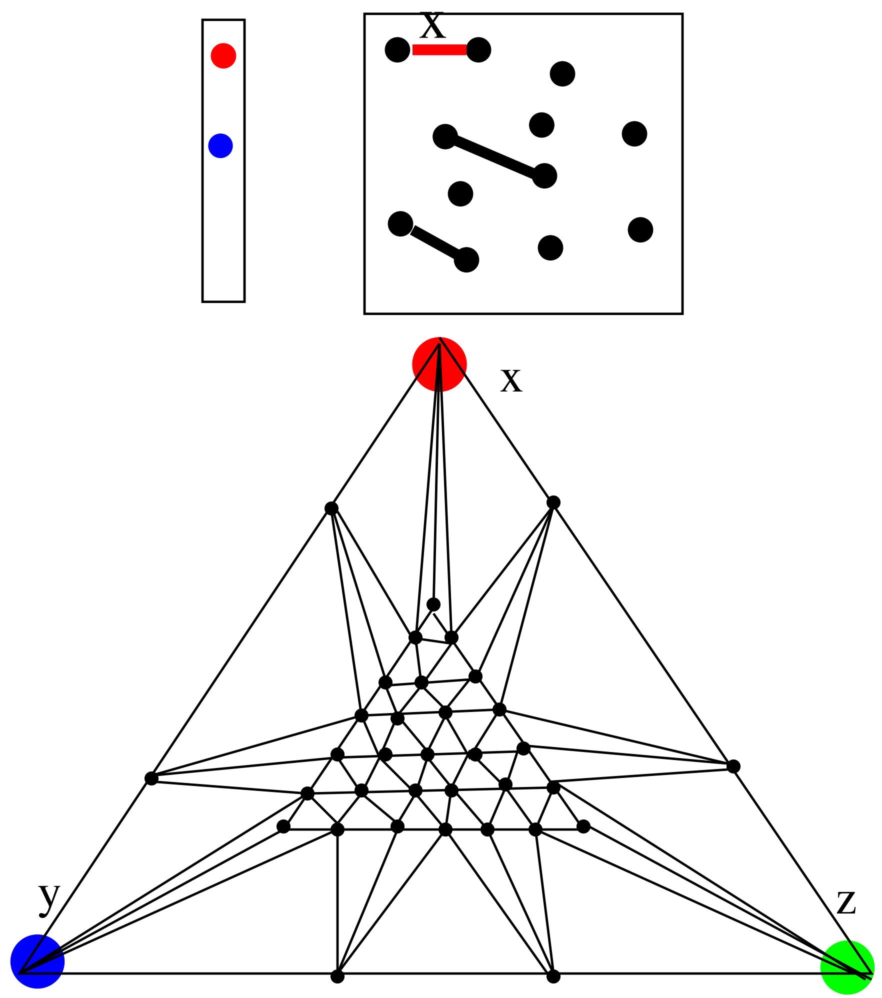
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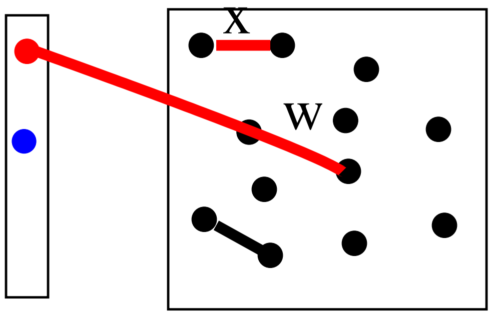

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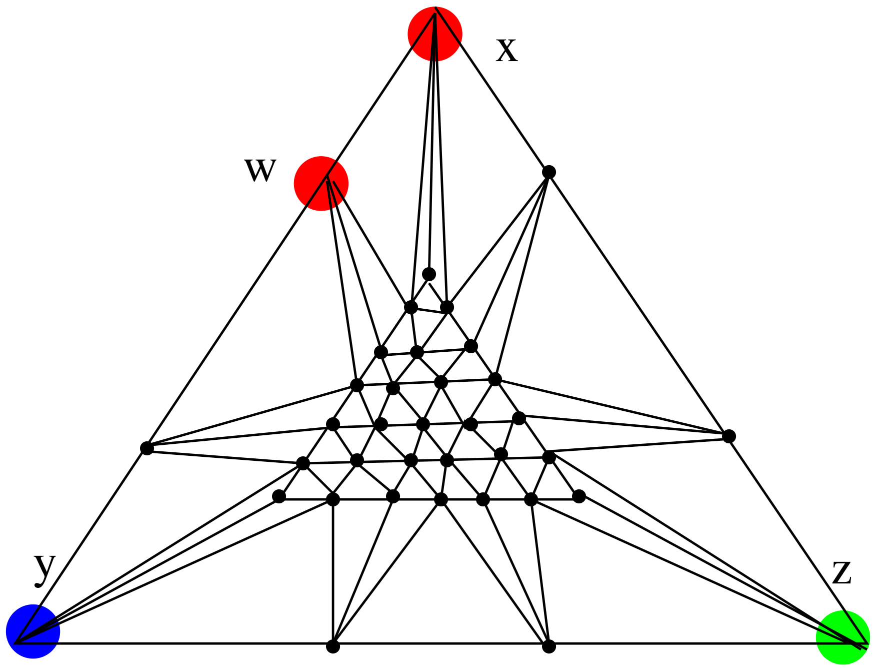

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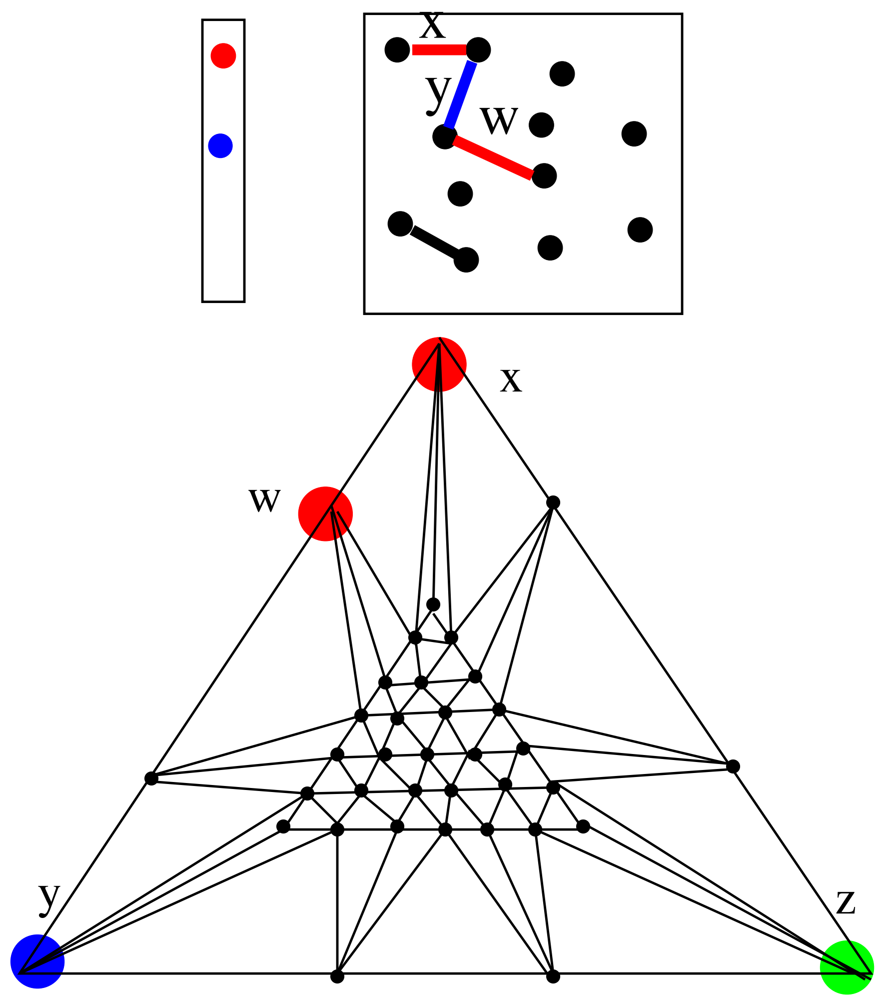
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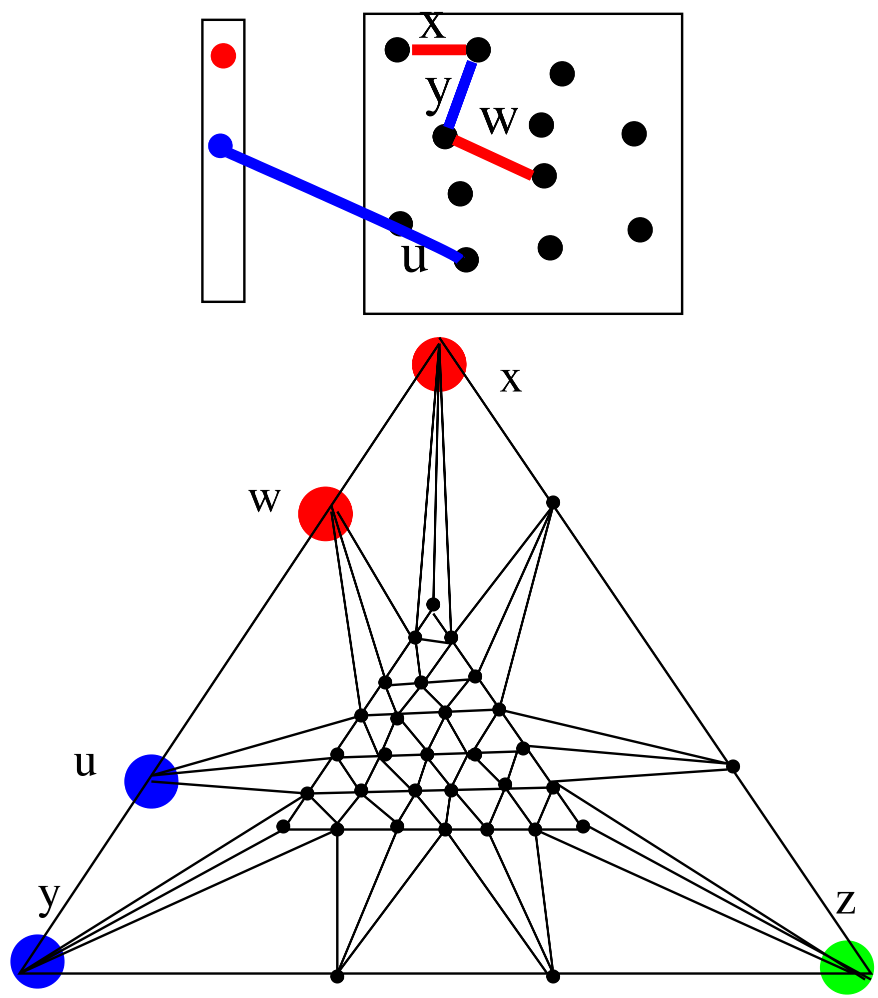
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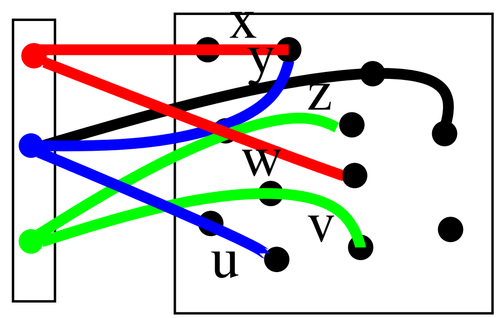
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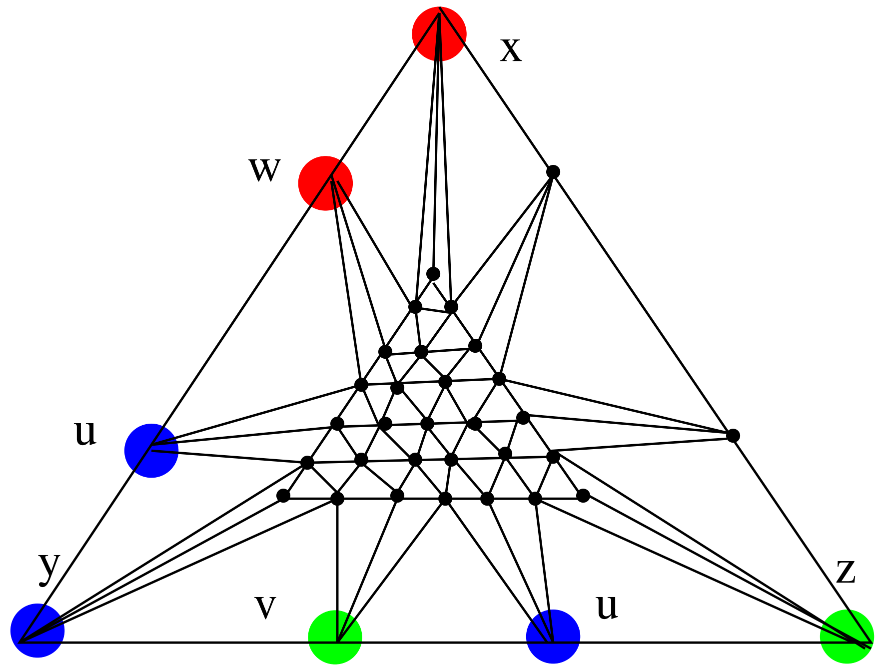
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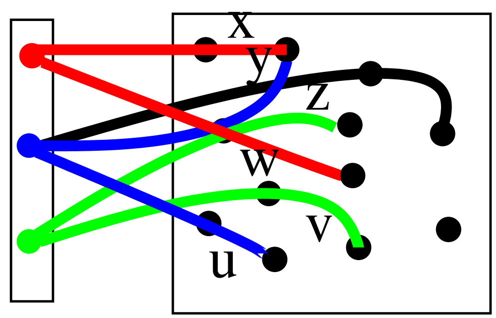
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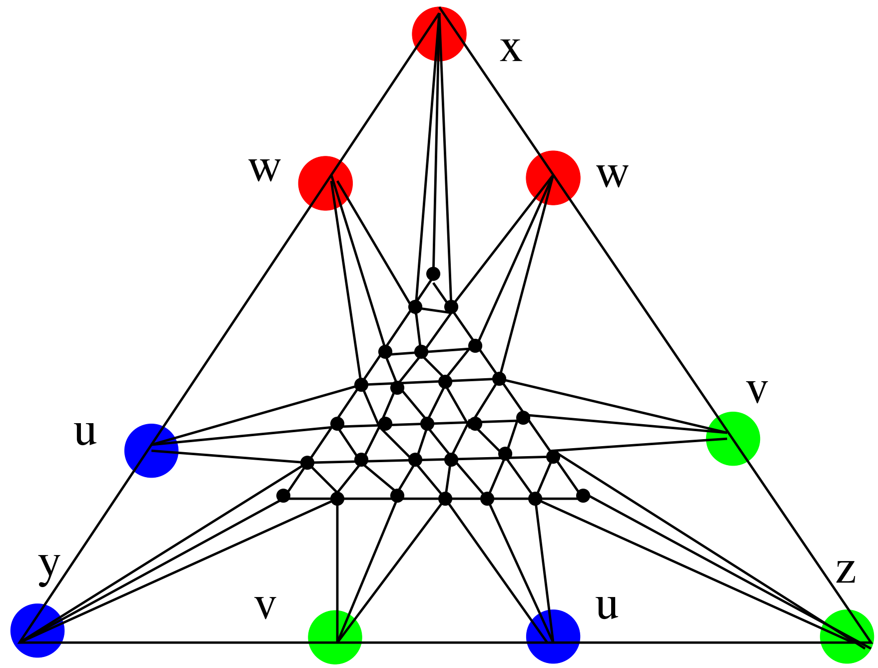
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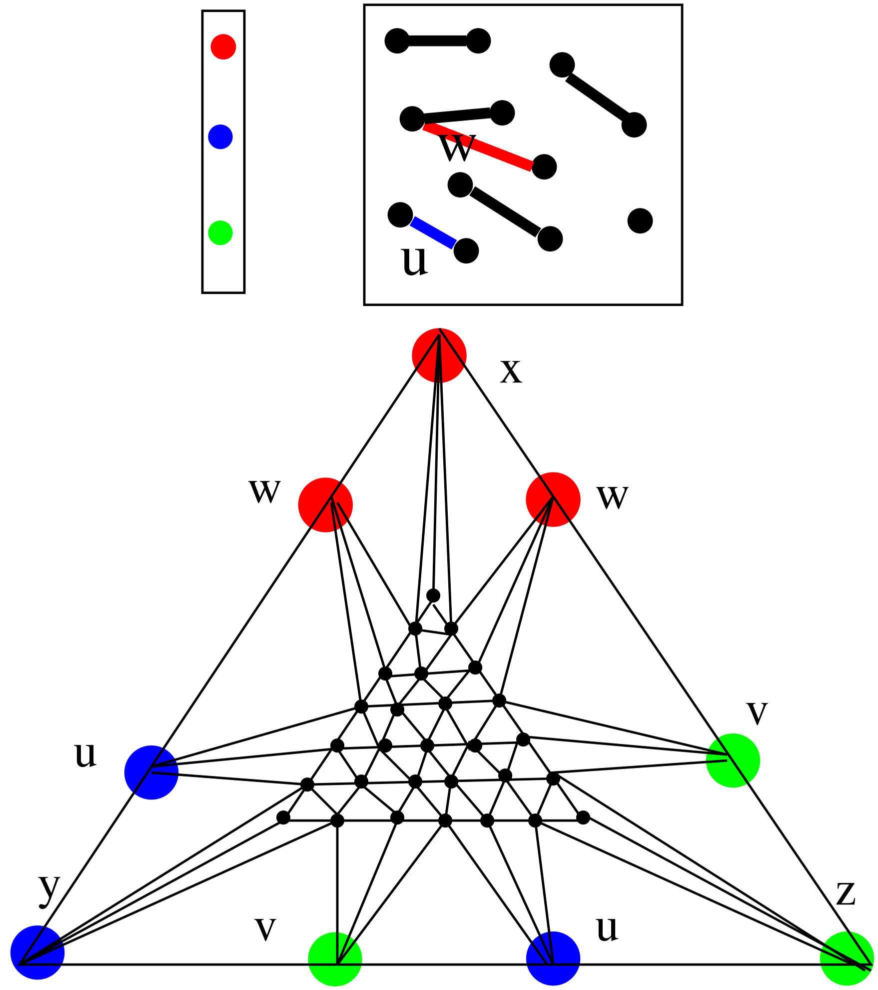
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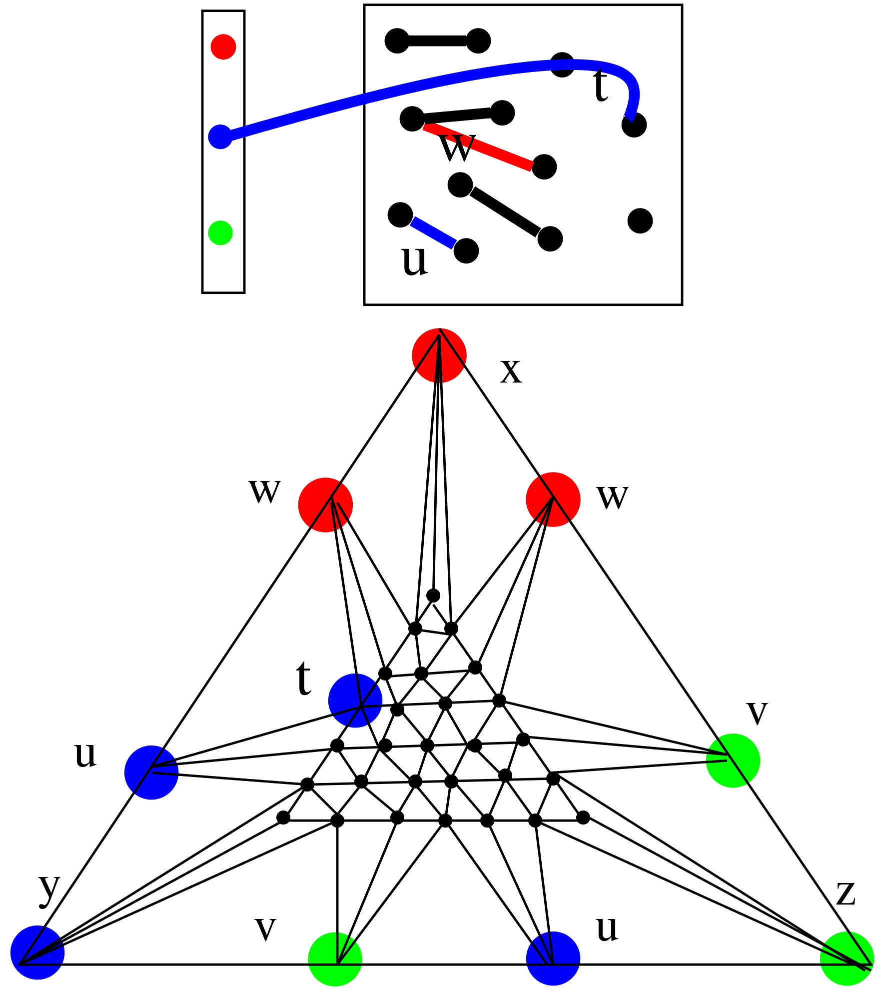
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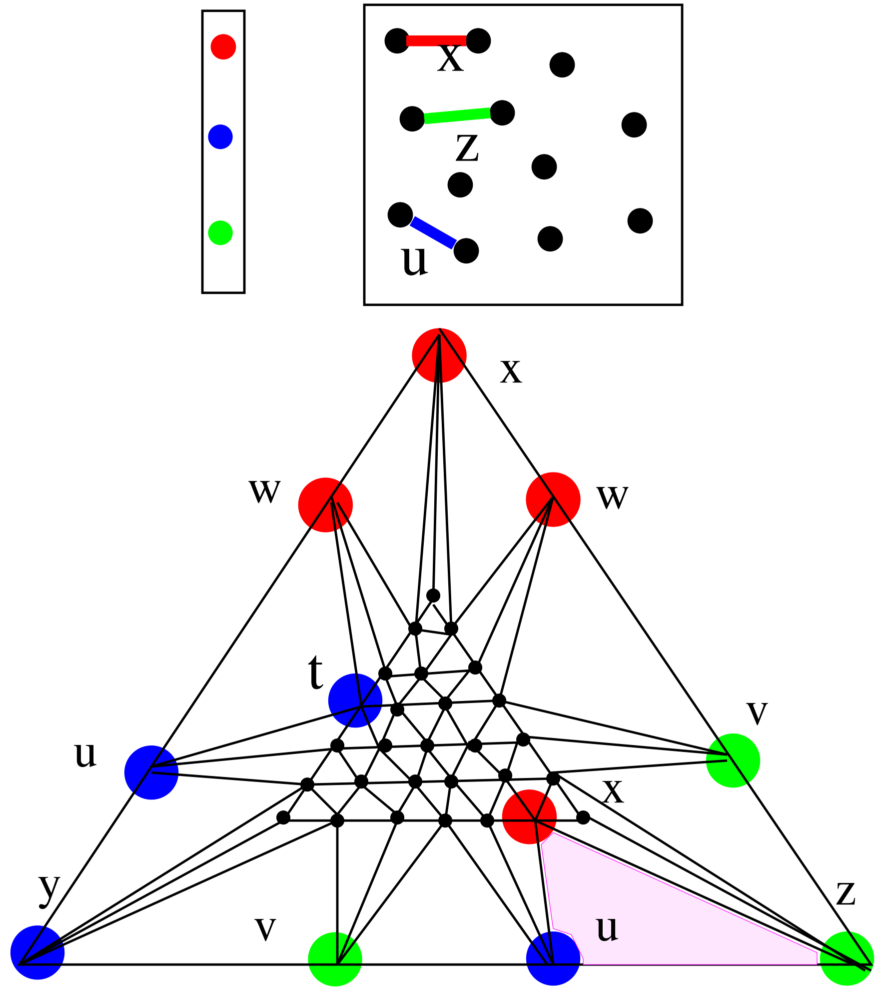
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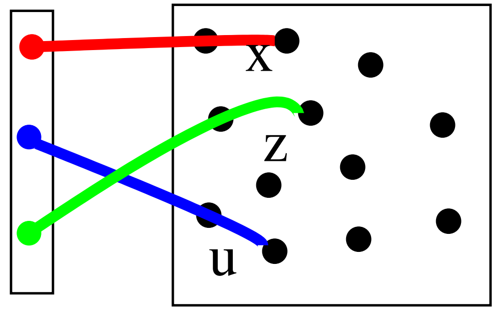
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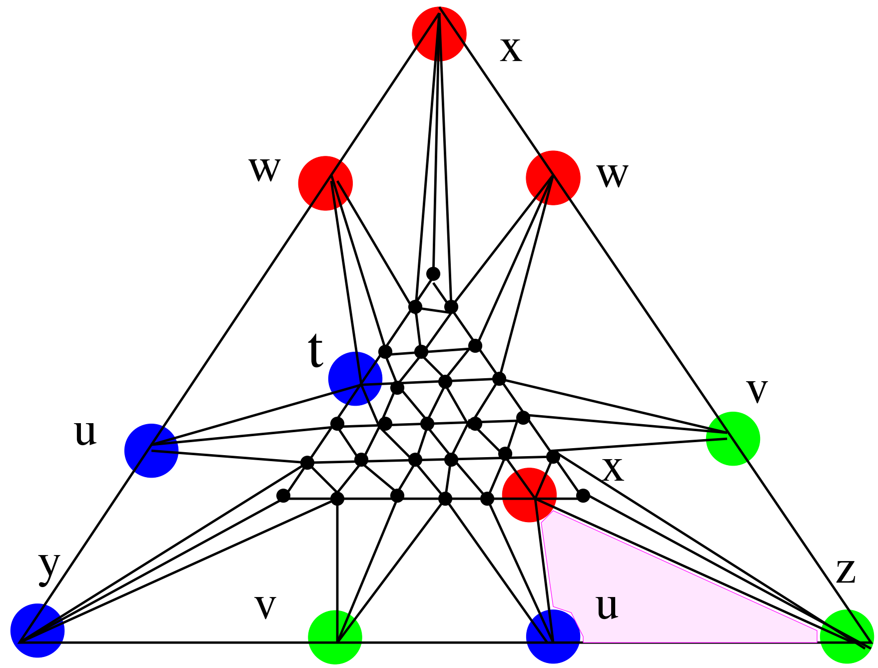
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When does a good committee exist?

- Not always.

- In the Unhappy Families case: when every subset S of departments contains representatives from at least $|S|$ families. (Hall's Theorem. Moreover a good committee can be found efficiently if it exists.)

- the Big Issues case: same as deciding the SAT problem. (So we cannot expect an efficient characterization.)

- if no faculty member conflicts with more than d others, and departments all have size at least $2d$.

- if for every $I \subset \{1, \ldots, m\}$ there exists an independent set S_I in $G_I = G[\cup_{i \in I} V_i]$ such that every independent set T in G_I of size at most $|I| - 1$ can be extended by a vertex of S_I.

Applications

- (hypergraph matching) P. Haxell, “A condition for matchability in hypergraphs”, *Graphs Comb.* 11 (1995), 245–248.
- (hypergraph matching) M. Krivelevich, “Almost-perfect matching in random uniform hypergraphs”, *Disc. Math.* 170 (1997), 259–263.
- (graph partitioning) N. Alon, G. Ding, B. Oporowski, D. Vertigan, “Partitioning into graphs with only small components”, *J Comb. Th B* 87 (2003), 231–243.
- (list colouring) P. Haxell, “A note on vertex list colouring”, *Comb. Prob. Comput.* 10 (2001), 345–348.

- (fractional colouring) R. Aharoni, E. Berger, R. Ziv, "Independent systems of representatives in weighted graphs", *Combinatorica* 27 (2007), 253–267
- (group theory) J. Britnell, A. Evseev, R. Guralnick, P. Holmes, A. Maróti, "Sets of elements that pairwise generate a linear group", *JCTA* 115 (2008), 442–465
- (group theory) A. Lucchini, A. Maróti, "On finite simple groups and Kneser graphs", *J. Alg. Comb.* 30 (2009), 549–566
- (circular colouring) T. Kaiser, D. Král, R. Skrekovski, "A revival of the girth conjecture", *JCTB* 92 (2004), 41–53

- (circular colouring) T. Kaiser, D. Král, R. Skrekovski, X. Zhu, “The circular chromatic index of graphs of high girth”, *JCTB* 97 (2007), 1–13

- (special transversals) L. Rabern, “On hitting all maximum cliques with an independent set”, *J. Graph Th.* 66 (2011), 32–37

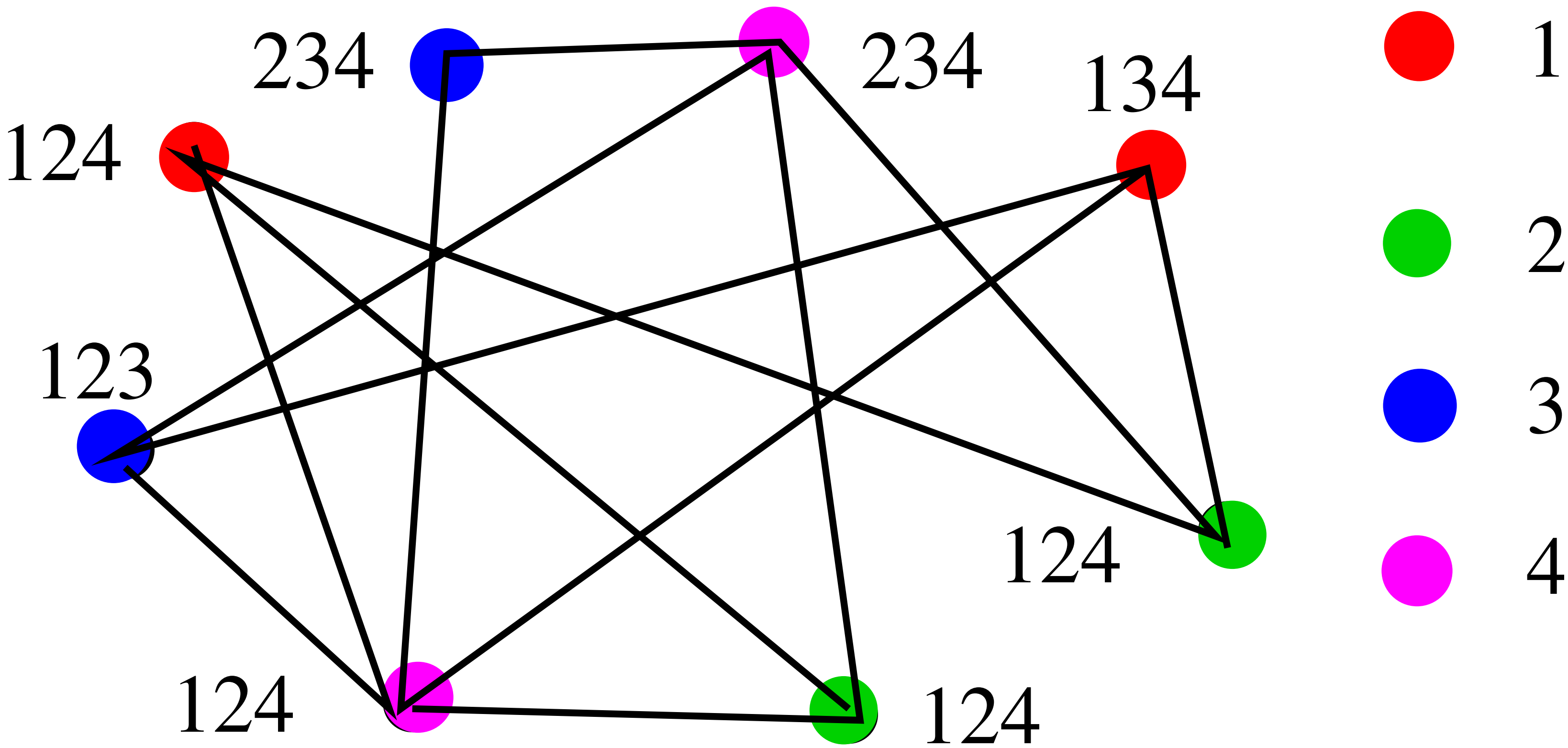
- (special transversals) A. King, “Hitting all maximum cliques with a stable set using lopsided independent transversals”, *J. Graph Th.* doi: 10.1002/jgt.20532

Applications: list colouring for graphs

Let G be a graph, with vertex set $V(G)$. Suppose each vertex v is assigned a list $L(v) \subset \{1, 2, \ldots\}$ of acceptable colours for v. An L-list colouring of G is a function $f : V(G) \to \{1, 2, \ldots\}$ such that

- whenever vertices x and y are joined by an edge, we have $f(x) \neq f(y)$, AND
- $f(v) \in L(v)$ for each $v \in V(G)$.

The smallest k for which EVERY list assignment L satisfying $|L(v)| \geq k$ for each v admits an L-list colouring is called the list chromatic number of G and is denoted $\chi_l(G)$.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References

- P.E. Haxell, A condition for matchability in hypergraphs, *Graphs and Combinatorics* 11 (1995), pp. 245-248
- P.E. Haxell, On the strong chromatic number, *Combinatorics, Probability and Computing* 13 (2004), pp. 857–865
- E. Sperner, Neuer Beweis für die Invarianz der Dimensionzahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* 6 (1928), 265–272.
- R. Aharoni and P. Haxell, Hall's Theorem for hypergraphs, *Journal of Graph Theory* 35 (2000), 83–88.