Combinatorics of polytopes and differential equations

Victor M. Buchstaber

Steklov Institute, RAS, Moscow
⟨buchstab@mi.ras.ru⟩

School of Mathematics, University of Manchester
⟨Victor.Buchstaber@manchester.ac.uk⟩

“BMS Friday Colloquium”

Berlin

05 February 2010

The talk is based on the papers:

1. V. M. Buchstaber,
Ring of Simple Polytopes and Differential Equations., Proceedings of the Steklov Institute of Mathematics, v. 263, 2008, 1–25.

2. V. M. Buchstaber, T. E. Panov,
Torus actions and their applications in topology and combinatorics., AMS, University Lecture Series, v. 24, Providence,RI, 2002.

3. V. M. Buchstaber,
Lectures on Toric Topology., Toric Topology Workshop, KAIST 2008, Trends in Mathematics, Information Center for Mathematical Sciences, V. 11, N. 1, 2008, 1–55.

4. V. M. Buchstaber, N. Yu. Erokhovets,
Ring of polytopes, quasisymmetric functions and Fibonacci numbers., in preparation.

Part I

Abstract

Polytopes are a classical object of convex geometry. They play a key role in many modern fields of research, such as algebraic and symplectic geometry, toric geometry and toric topology, enumerative combinatorics, and mathematical physics.

We describe the results of a new approach based on a differential ring of combinatorial polytopes.

This approach allows to apply the theory of differential equations to the study of polytopes.

As an application we consider the differential subrings of nestohedra and describe explicitly the generating functions of important families of graph-associahedra.

Contents

Basic definitions

Let us consider the n-dimensional Euclidean space $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is $x = (x_1, \dots, x_n)$, where $x_k \in \mathbb{R}$, $k = 1, \dots, n$, is a real number.

Definition 1. A ***convex hull*** of a finite set $\{v_1, \dots, v_N\}$ of points in $\mathbb{R}^n$ is

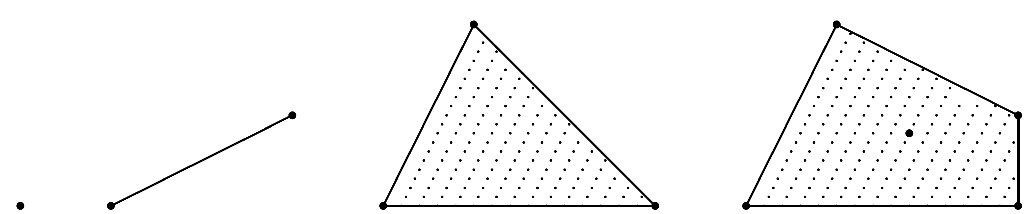
$$\mathrm{conv}(v_1, \dots, v_N) = \left\{ x \in \mathbb{R}^n \ : \ x = \sum_{i=1}^{N} t_i v_i,\ t_i \geqslant 0,\ \sum_{i=1}^{N} t_i = 1 \right\}.$$

Definition 2. For some set $\{v_1, \dots, v_N\}$ of points a ***convex polytope*** in $\mathbb{R}^n$ is

$$P = \mathrm{conv}(v_1, \dots, v_N).$$

We will speak about polytopes without including the word "convex".

Example. Polytopes in $\mathbb{R}^2$



$N = 1$ $N = 2$ $N = 3$ $N = 5$

Definition 3. An n-dim *convex polyhedron* P is an intersection of finitely many half-spaces in $\mathbb{R}^n$:

$$P = \{x \in \mathbb{R}^n \ : \ \langle l_i, x\rangle + a_i \geqslant 0, \ i = 1, \dots, m\}, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product in $\mathbb{R}^n$ and $l_i \in \mathbb{R}^n$, $a_i \in \mathbb{R}$, $i = 1, \dots, m$.

A polytope is a *bounded convex polyhedron*.

Agreement. Suppose that a polytope P^n is represented as an intersection of half-spaces as in (1).

In the sequel we assume that there are no redundant inequalities $\langle l_i, x\rangle + a_i \geqslant 0$ in such a representation.

That is, no inequality can be removed from (1) without changing the polytope P^n.

In this case P^n has exactly m facets which are the intersections of the hyperplanes $\langle l_i, x\rangle + a_i = 0$, $i = 1, \dots, m$, with P^n.

The vector l_i is orthogonal to the corresponding facet and *points towards the interior* of the polytope.

Definitions 2 and 3 produce the same geometrical object, i.e. a subset of $\mathbb{R}^n$ is a convex hull of a finite point set if and only if it is a bounded intersection of finitely many half-spaces.

The notion of *generic polytope* depends on the choice of definition of convex polytope.

A set of $m > n$ points in $\mathbb{R}^n$ is in *general position* if no $(n + 1)$ of them lie in a common affine hyperplane. Now Definition 2 suggests to call a *convex polytope generic* if it is the convex hull of a set of *general positioned points*.

This implies that all proper faces of the polytope are simplices, i.e. every facet has the minimal number of vertices (namely, n).

Such polytopes are called *simplicial*.

On the other hand, a set of $m > n$ hyperplanes

$\langle l_i, x\rangle + a_i = 0,\ l_i \in \mathbb{R}^n,\ x \in \mathbb{R}^n,\ a_i \in \mathbb{R},\ i = 1, \ldots, m,$

is in *general position* if no point belongs to more than n hyperplanes.

From the viewpoint of Definition 3, a convex polytope P^n is *generic* if its bounding *hyperplanes* are in *general position*.

That is, there are exactly n facets meeting at each vertex of P^n. Such polytopes are called *simple*. Note that each face of a simple polytope is again a simple polytope.

Differential ring of combinatorial polytopes

Definition. Two polytopes P_1 and P_2 of the same dimension are said to be *combinatorially equivalent* if there is a bijection between their sets of faces that preserves the inclusion relation.

Definition. A *combinatorial polytope* is a class of combinatorial equivalent polytopes.

Denote by $\mathscr{P}^{2n}$ the free abelian group generated by all n-dimensional combinatorial polytopes.

For $n \geqslant 1$ we have the direct sum

$$\mathscr{P}^{2n} = \sum_{m \geqslant n+1} \mathscr{P}^{2n,2(m-n)},$$

where $P^n \in \mathscr{P}^{2n,2(m-n)}$ if it is a polytope with m facets and $\operatorname{rank} \mathscr{P}^{2n,2(m-n)} < \infty$ for any fixed n and m.

Definition. The product of polytopes turns the direct sum

$$\mathscr{P} = \sum_{n \geqslant 0} \mathscr{P}^{2n} = \mathscr{P}^0 + \sum_{m \geqslant 2} \sum_{n=1}^{m-1} \mathscr{P}^{2n,2(m-n)}$$

into a bigraded commutative associative ring, the *ring of polytopes*. The unit is P^0, a point.

The direct product $P_1^n \times P_2^m$ of simple polytopes P_1^n and P_2^m is a simple polytope as well.

Thus the ring $\mathscr{P}_s$ generated by simple polytopes is a subring in $\mathscr{P}$.

A polytope is *indecomposable* if it can not be represented as a product of two other polytopes of positive dimension.

Theorem. The ring $\mathscr{P}$ is a polynomial ring generated by indecomposable combinatorial polytopes.

Let P^n be a polytope. Denote by dP^n the *disjoint union* of all its facets.

Lemma. There is a linear operator of degree -2

$$d : \mathscr{P} \longrightarrow \mathscr{P},$$

such that

$$d(P_1^{n_1} P_2^{n_2}) = (dP_1^{n_1})P_2^{n_2} + P_1^{n_1}(dP_2^{n_2}).$$

Thus, $\mathscr{P}$ is a *differential ring*, and $\mathscr{P}_S$ is a *differential subring* in $\mathscr{P}$.

Examples:

$$dI^n = n(dI)I^{n-1} = 2nI^{n-1},$$

$$d\Delta^n = (n+1)\Delta^{n-1},$$

where Δ^n is the standard n-simplex and $I^n = I \times \cdots \times I$ is the standard n-cube.

f-polynomial (face-polynomial)

Consider the linear map

$$f\colon \mathscr{P} \longrightarrow \mathbb{Z}[\alpha, t],$$

which sends a polytope P^n to the homogeneous *face-polynomial*

$$f(P^n) = \alpha^n + f_{n-1,1}\alpha^{n-1}t + \cdots + f_{1,n-1}\alpha t^{n-1} + f_{0,n}t^n,$$

where $f_{k,n-k} = f_{k,n-k}(P^n)$ is the number of its k-dim faces.

Thus $f_{n-1,1}$ is the number of facets and $f_{0,n}$ is the number of vertices.

Theorem.
1. The mapping f is a *ring homomorphism*.
2. Let P be a polytope, then

$$f(dP) = \frac{\partial}{\partial t} f(P)$$

if and only if P is *simple*.

Theorem. Let $\widehat{f}\colon \mathscr{P}_s \to \mathbb{Z}[t, \alpha]$ be a *linear* map such that

$$\widehat{f}(dP^n) = \frac{\partial}{\partial t}\widehat{f}(P^n) \text{ and } \widehat{f}(P^n)|_{t=0} = \alpha^n.$$

Then $\widehat{f}(P^n) = f(P^n)$.

Dehn–Sommerville relations

Theorem. For any *simple polytope* P^n we have

$$f(P^n)(\alpha, t) = f(P^n)(-\alpha, \alpha + t).$$

Proof. We have

$$f(P^0)(\alpha, t) = 1 = f(P^0)(-\alpha, \alpha + t).$$

By induction let it be true for all $k \leqslant n$. Then

$$f(dP^{n+1})(\alpha, t) = f(dP^{n+1})(-\alpha, \alpha + t).$$

Thus

$$\frac{\partial}{\partial t} f(P^{n+1})(\alpha, t) = \frac{\partial}{\partial t} f(P^{n+1})(-\alpha, \alpha + t).$$

Hence,

$$f(P^{n+1})(\alpha, t) - f(P^{n+1})(-\alpha, \alpha + t) = c(\alpha).$$

The simple polytope P^{n+1} has the canonical structure of a cellular complex, where faces are cells. Thus,

$$f(-\alpha,\alpha) = \Big((-1)^{n+1} + (-1)^n f_{n,1} + \cdots + f_{0,n+1}\Big)\alpha^{n+1} =$$
$$= \chi(P^{n+1})\alpha^{n+1} = \alpha^{n+1}.$$

Here $\chi(P^{n+1})$ is the Euler characteristic of P^{n+1}. Therefore,

$$c(\alpha) = f(P^{n+1})(\alpha,0) - f(P^{n+1})(-\alpha,\alpha) = 0.$$

The Dehn–Sommerville relations were established by Dehn for $n \leqslant 5$ in 1905 and by Sommerville in the general case in 1927 in the form of equations

$$f_{k,n-k} = \sum_{j=0}^{k}(-1)^j\binom{n-j}{k-j}f_{j,n-j}$$

which are equivalent to the formula

$$f(P^n)(\alpha,t) = f(P^n)(-\alpha,\alpha+t).$$

h-polynomial (height-polynomial)

Set $h(P^n)(\alpha,t) = \alpha^n + h_1\alpha^{n-1}t + \cdots + h_{n-1}\alpha t^{n-1} + t^n$, where $h(P^n)(\alpha,t) = f(P^n)(\alpha - t, t)$.

From Dehn–Sommerville relations we obtain

$$h(P^n)(\alpha,t) = h(P^n)(t,\alpha).$$

For example,

$$h(I^n)(\alpha,t) = (\alpha+t)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} t^k,$$

$$h(\Delta^n)(\alpha,t) = \frac{\alpha^{n+1} - t^{n+1}}{\alpha - t} = \sum_{k=0}^{n} \alpha^{n-k} t^k.$$

Corollary. Set $\partial = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial t}$.

1. The mapping $h\colon \mathscr{P} \to Z[\alpha,t]$ is the ring homomorphism such that $h(P^n)(\alpha,0) = \alpha^n$.

2. $h(dP^n) = \partial h(P^n)$ if and only if P^n is simple.

3. Let $\widehat{h}\colon \mathscr{P}_s \longrightarrow Z[\alpha,t]$ be a *linear* mapping such that

$$\widehat{h}(dP^n) = \partial\widehat{h}(P^n),\ \widehat{h}(P^n)(\alpha,0) = \alpha^n.$$

Then $\widehat{h}(P^n) = h(P^n)$.

The Dehn–Sommerville relations did not become well-known until V. Klee reproved them in a more general context and obtained the following result.

Proposition. (Klee, 1964)
The Dehn–Sommerville relations are the most general linear relations satisfied by f-vectors of all simple polytopes.

Proof. Set $Q_k = \Delta^k \times \Delta^{n-k},\ k = 0, 1 \ldots, [\frac{n}{2}]$.
We have

$$h(Q_k) = \frac{\alpha^{k+1} - t^{k+1}}{\alpha - t} \cdot \frac{\alpha^{n-k+1} - t^{n-k+1}}{\alpha - t}$$

and $h(Q_{k+1}) - h(Q_k) = \alpha^{n-k-1}t^{k+1} + \ldots + \alpha^{k+1}t^{n-k-1}$.
Therefore the polynomials $h(Q_k),\ k = 0, 1 \ldots, [\frac{n}{2}]$, are affinely independent. $\square$

In the Klee's paper this statement was proved directly in terms of f-vectors. The usage of the ring homomorphism $h\colon \mathscr{P}_s \to \mathbb{Z}[\alpha, t]$ significantly simplifies the proof.

Minkowski sum

Let M_1 and M_2 be subsets in $\mathbb{R}^n$.

Definition. A *Minkowski sum* of M_1 and M_2 is the set
$$\{x \in \mathbb{R}^n \,:\, x = x_1 + x_2,\ x_1 \in M_1,\ x_2 \in M_2\}.$$

Lemma. If M_1 and M_2 are convex polytopes then $M_1 + M_2$ is again a convex polytope.

The collection of all convex polytopes in $\mathbb{R}^n$ is denoted by $\mathscr{M}_n$.

The Minkowski sum gives an abelian monoid structure on $\mathscr{M}_n$, where zero 0 is the point $0 = (0, \ldots, 0) \in \mathbb{R}^n$.

Proposition. Minkowski sum of two polytopes is a polytope. Moreover, if $P = conv(v_1, \ldots, v_k)$ and $Q = conv(w_1, \ldots, w_l)$, then
$$P + Q = conv(v_1 + w_1, \ldots, v_i + w_j, \ldots, v_k + w_l).$$

Minkowski sum of simplices

Let $e_i,\ i = 1, \ldots, n+1$, be the endpoints of the standard basis vectors in $\mathbb{R}^{n+1}$.

The Minkowski sum of the segments $[0, e_i],\ i = 1, 2, 3$, in $\mathbb{R}^3$ is the standard cube I^3.

More generally, in $\mathbb{R}^n$ the Minkowski sum of line segments forms a polytope known as a *zonotope*.

The Minkowski sum of four edges of an octahedron with a common vertex is a *rhombic dodecahedron*.
It is a convex polyhedron with 12 rhombic faces,
24 edges and 14 vertices.

Some minerals such as garnet form a rhombic dodecahedral crystal habit. Honeybees use the geometry of rhombic dodecahedra to form honeycomb.

It gives an example when a Minkowski sum of the simple polytopes forms a nonsimple polytope.

For every subset $S \subset [n+1] = \{1, \dots, n+1\}$ consider the regular simplex

$$\Delta_S = \operatorname{conv}(e_i \colon i \in S) \subset \mathbb{R}^{n+1}.$$

Let $\mathscr{F}$ be a collection of subsets S of $[n+1]$.
We assume that $\mathscr{F}$ contains all singletons $\{i\}$, $1 \leqslant i \leqslant n+1$.

Consider the convex polytope

$$P_{\mathscr{F}} = \sum_{S \in \mathscr{F}} \Delta_S \subset \mathbb{R}^{n+1}.$$

As usual, denote by $|\mathscr{F}|$ the number of elements in $\mathscr{F}$.

Proposition. (E.-M. Feichtner, B. Sturmfels, 2005)
$P_{\mathscr{F}}$ can be described as the intersection of the hyperplane

$$H_{\mathscr{F}} = \{x \in \mathbb{R}^{n+1} \colon \sum_{i=1}^{n+1} x_i = |\mathscr{F}|\}$$

with the halfspaces

$$H_{T,\geqslant} = \{x \in \mathbb{R}^{n+1} \colon \sum_{i \in T} x_i \geqslant \left|\mathscr{F}|_T\right|\}$$

corresponding to all subsets $T \subset [n+1]$.

Building sets

Definition. A collection B of non-empty subsets of the set $[n+1] = \{1, \ldots, n+1\}$ is called a *building set* if:

- $S', S'' \in B$ and $S' \cap S'' \neq \emptyset \Rightarrow S' \cup S'' \in B$,
- $\{i\} \in B$ for all $i \in [n+1]$.

A building set B on $[n+1]$ is said to be *connected* if $[n+1] \in B$.

Theorem. (A. Postnikov, 2005, E.-M. Feichtner, B. Sturmfels, 2005) P_B can be described as the intersection of the hyperplane

$$P_B = \left\{x \in \mathbb{R}^{n+1} \colon \textstyle\sum_{i=1}^{n+1} x_i = |B|\right\}$$

with the halfspaces

$$H_S = \left\{x \in \mathbb{R}^{n+1} \colon \sum_{i \in S} x_i \geqslant \Big| B|_S \Big| \quad \text{for every } S \in B\right\}.$$

If B is connected, then this representation is irredundant, that is, every hyperplane ∂H_S with $S \neq [n+1]$ defines a facet F_S of P_B
(so there are $|B| - 1$ facets in total).

Theorem. The intersection of facets $F_{S_1} \cap \ldots \cap F_{S_k}$ is nonempty (and therefore gives a face of P_B) if and only if the following two conditions are satisfied:

(a) for any $i, j,\ \ 1 \leqslant i < j \leqslant k$, either $S_i \subset S_j$, or $S_j \subset S_i$, or $S_i \cap S_j = \varnothing$;

(b) if the sets $S_{i_1}, \ldots, S_{i_k}$ are pairwise nonintersecting, then $S_{i_1} \cup \ldots \cup S_{i_k} \notin B$.

Definition. For any building set B the polytope P_B is called a *nestohedron*.

Corollary. The nestohedron P_B is a simple polytope.

Definition. Let Γ be a simple graph (no loops, no multiple edges) with vertex set $[n{+}1] = \{1, \ldots, n{+}1\}$. A *graphical building set* $B(\Gamma)$ is the set of all non-empty subsets $S \subset [n+1]$ such that the graph $\Gamma|_S$ is connected.

Lemma. The graphical building set $B(\Gamma)$ is a building set.

Definition. For any simple graph Γ the polytope $P_{B(\Gamma)}$ is called a *graph-associahedron*.

Ring of building sets

Definition. Let $B_i,\ i = 1, 2$, be the building sets on $[n_i + 1]$. A map

$$\xi\colon (B_1, [n_1 + 1]) \longrightarrow (B_2, [n_2 + 1])$$

of the building sets is a map

$$\xi\colon [n_1 + 1] \longrightarrow [n_2 + 1]$$

such that $\xi^{-1}(S) \in B_1$ for any $S \in B_2$.

Two building sets B_1 and B_2 on $[n + 1]$ are said to be *equivalent*, if there exists a permutation σ of $[n + 1]$ such that σ defines a map $B_1 \to B_2$ and σ^{-1} defines a map $B_2 \to B_1$.

Denote by $\mathscr{B}^n$ the abelian group generated by the equivalence classes of building sets on $[k + 1],\ k \leqslant n$. Define the product of building sets B_l on $[n_l + 1], l = 1, 2$, as the building set $B = B_1 \cdot B_2$ on $[n_1 + n_2 + 2]$ induced by joining the interval $[n_1 + 1]$ to the interval $[n_2 + 1]$.

Thus we introduce the structure of a commutative associative ring $\mathscr{B}$ on the set $\bigcup \mathscr{B}^n$.

If B is a connected building set on $[n+1]$, then P_B is an *n-dimensional* simple polytope in $\mathbb{R}^{n+1}$.

The ring $\mathscr{B}$ is multiplicatively generated by connected building sets.

Let B be a building set on $[n+1]$, and let $S \in B$. For every $S \subset [n+1]$, we set

$$B|_S = \{S' \in B;\ \ S' \subseteq S\}$$

$$B/S = \{S' \subset [n+1]\backslash S;\ \ S' \in B \text{ or } S' \cup S \in B\}.$$

If B is a connected building set on $[n+1]$, then $B|_S$ is a connected building set on $|S|$ and B/S is a connected building set on $[n+1-|S|]$ for any $S \in B$, $S \neq [n+1]$.

Consider the linear mapping $d\colon \mathscr{B} \to \mathscr{B}$ defined as

$$dB = \sum_{S \in B \setminus [n+1]} B|_S \cdot B/S,$$

if B is a connected building set on $[n+1]$, and extended to the whole ring $\mathscr{B}$ by the Leibnitz law

$$d(B_1 \cdot B_2) = (dB_1) \cdot B_2 + B_1 \cdot (dB_2).$$

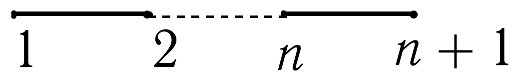
Theorem. The correspondence $B \to P_B$ defines the ring homomorphism $\beta\colon \mathscr{B} \to \mathscr{P}$ such that $\beta(dB) = d\beta(B)$.

The graphical building sets $P_{B(\Gamma)}$ generate the differential subring $\mathscr{T} \subset \mathscr{B}$.

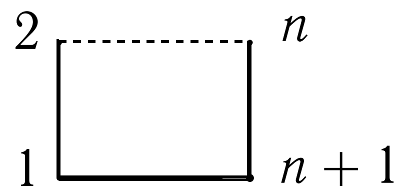
Graph-associahedra

Let Γ be a finite simple graph.
When Γ is:

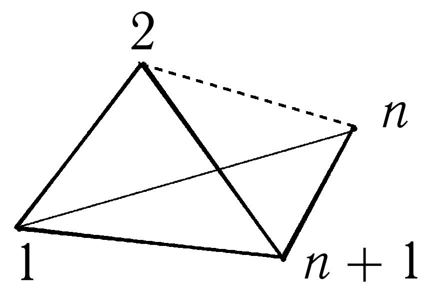
a path



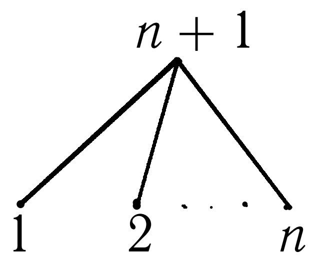
a cycle



a complete graph



an n-star graph

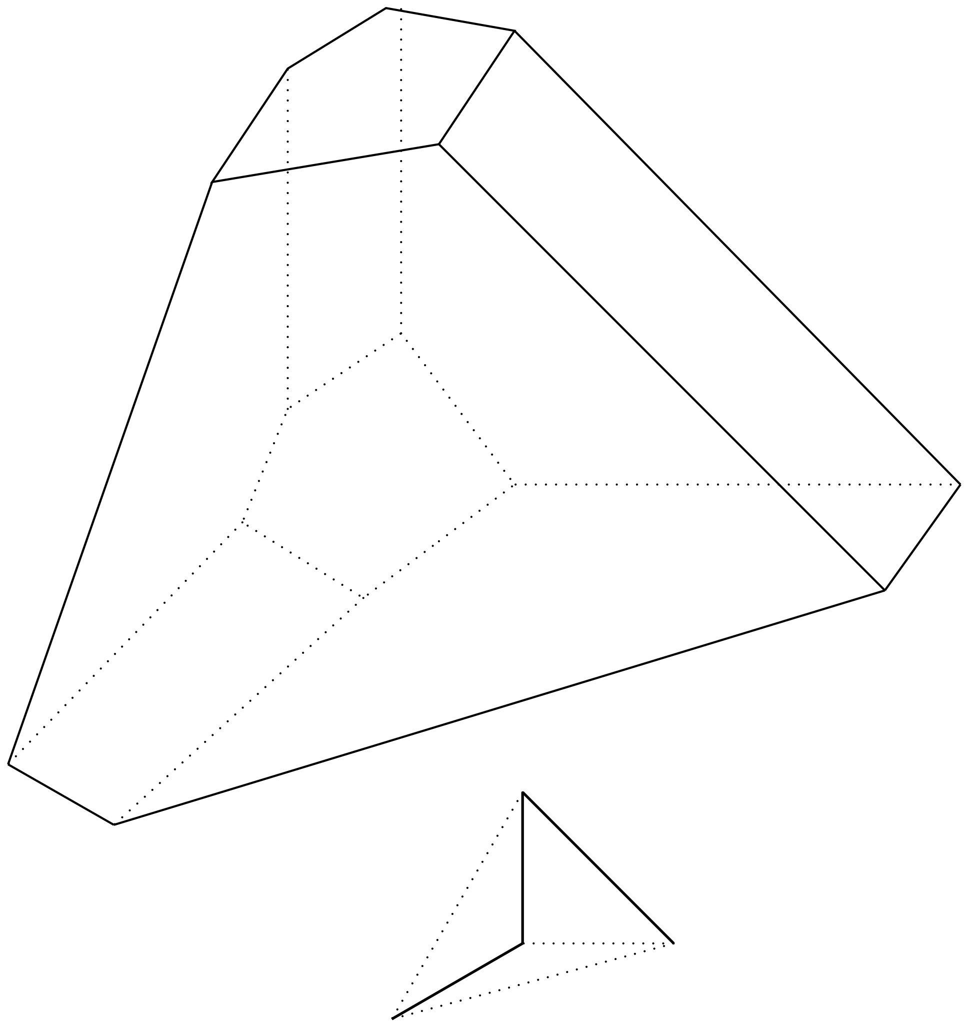


the polytope $P_{B(\Gamma)}$ results in the:

- associahedron (Stasheff polytope) As^n,
- cyclohedron (Bott–Taubes polytope) Cy^n,
- permutohedron Pe^n,
- stellohedron St^n, respectively.

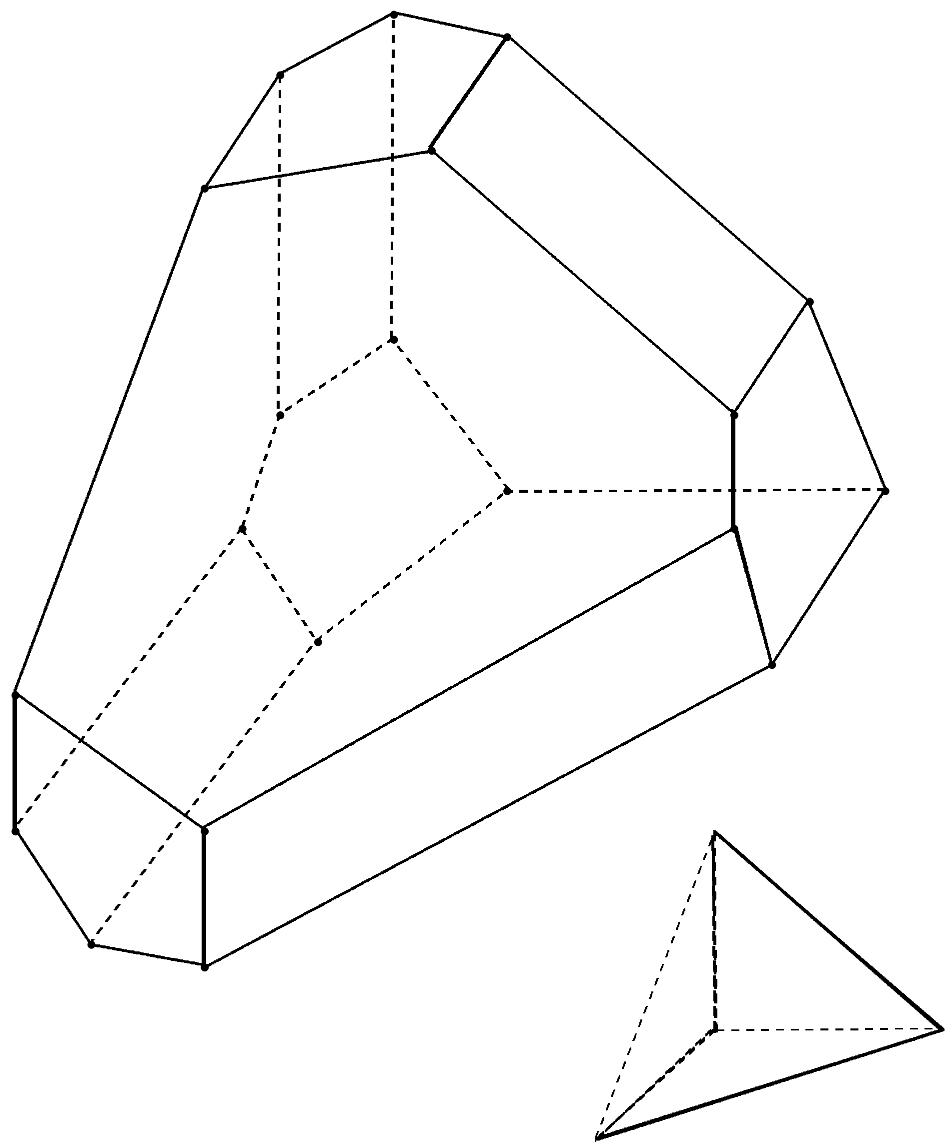
$As^2 = St^2$ is a 5 gon and $Cy^2 = Pe^2$ is a 6 gon.

Let Γ be a path with n edges $\{i, i+1\}$ for $1 \leqslant i \leqslant n$. Then $B(\Gamma)$ consists of all segments of the form $[i, j] = \{i, i+1, \ldots, j\}$ where $1 \leqslant i \leqslant j \leqslant n+1$, and $P_{B(\Gamma)}$ is associahedron As^n.



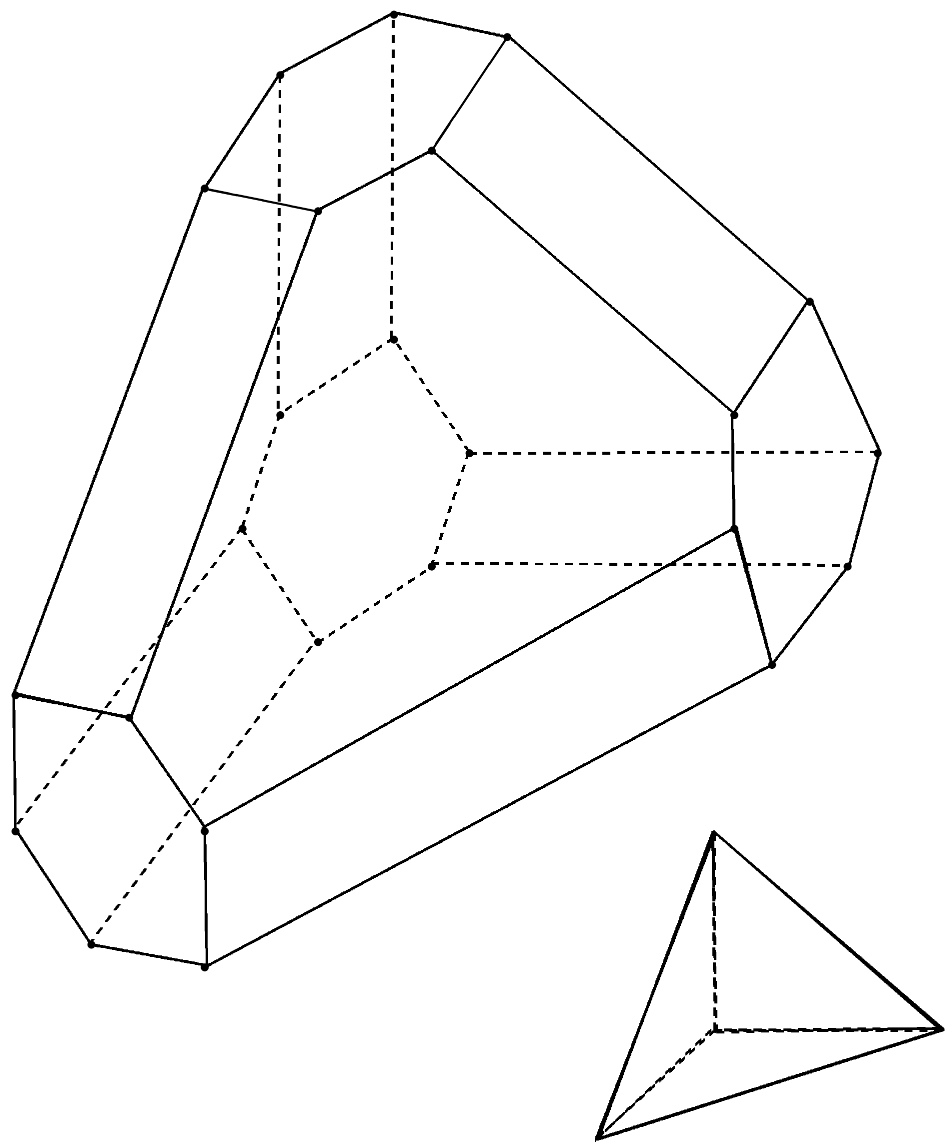
Associahedron As^3 and 3-path.

Let Γ be a cycle consisting of $n+1$ edges $\{i, i+1\}$ for $1 \leqslant i \leqslant n$ and $\{n+1, 1\}$. The corresponding $P_{B(\Gamma)}$ is known as the cyclohedron Cy^n or *Bott–Taubes polytope*.



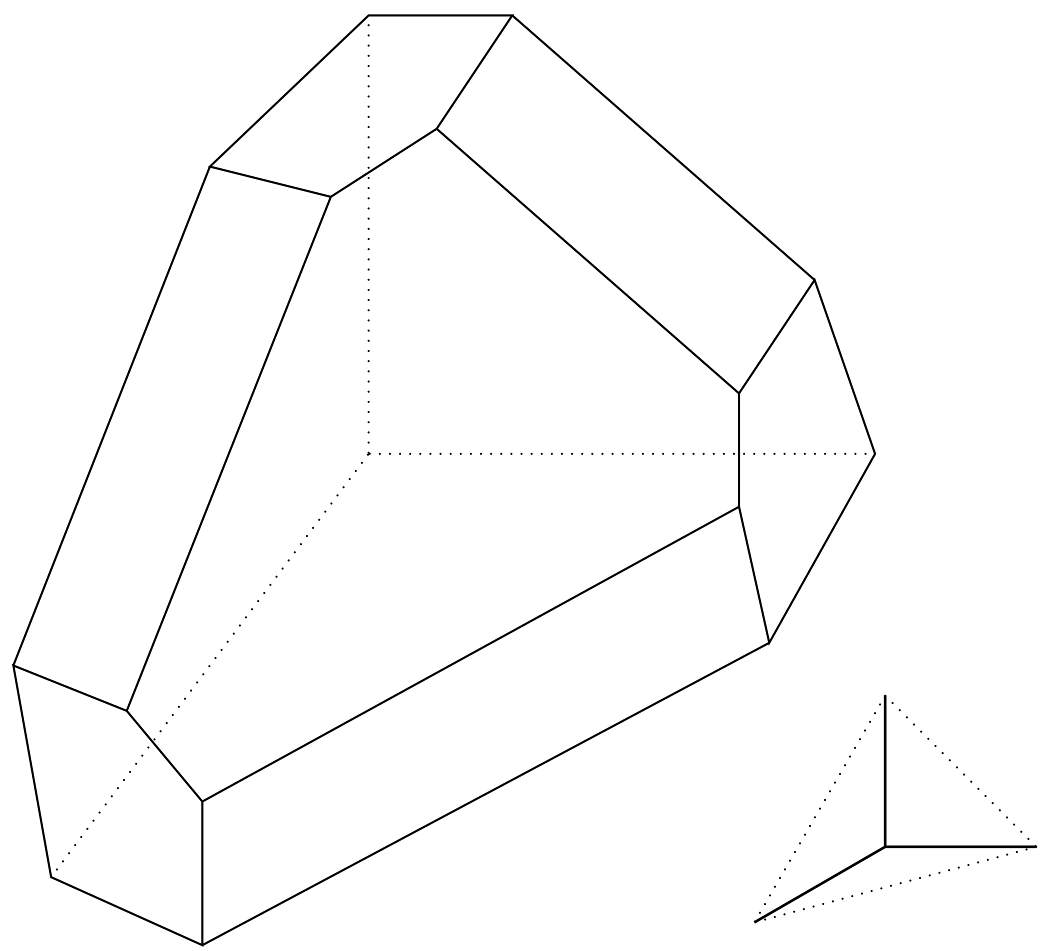
Cyclohedron Cy^3 and the corresponding graph

Let Γ be a complete graph; then $B(\Gamma)$ is the complete building set on $[n+1]$ and $P_{B(\Gamma)}$ is the permutohedron Pe^n.



Permutohedron Pe^3 and the corresponding graph

Let Γ be a star consisting of n edges $\{i, n+1\}$, $1 \leqslant i \leqslant n$, emanating from one point. The corresponding $P_{B(\Gamma)}$ is known as the stellohedron St^n



Stellohedron St^3 and the corresponding 3-star graph

Using the general formula for dB, one can obtain the explicit formulas for $dP_{B(\Gamma)}$:

$$dAs^n = \sum_{i+j=n-1} (i+2)As^i \times As^j$$

$$dCy^n = (n+1) \sum_{i+j=n-1} As^i \times Cy^j$$

$$dPe^n = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j$$

$$dSt^n = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}$$

For example (see pictures),

$$dAs^3 = 2As^0 \times As^2 + 3As^1 \times As^1 + 4As^2 \times As^0 \ ;$$
$$dCy^3 = 4(As^0 \times Cy^2 + As^1 \times Cy^1 + As^2 \times Cy^0) \ ;$$
$$dPe^3 = 4Pe^0 \times Pe^2 + 6Pe^1 \times Pe^1 + 4Pe^2 \times Pe^0 k \ ;$$
$$dSt^3 = 3St^2 + St^0 \times Pe^2 + 3St^1 \times Pe^1 + 3St^2 \times Pe^0 \ .$$

Families of polytopes and differential equations

We consider the following generating series of the six sequences of nestohedra:

$$\Delta(x) = \sum_{n \geqslant 0} \Delta^n \frac{x^{n+1}}{(n+1)!} \ ;$$

$$I(x) = \sum_{n \geqslant 0} I^n \frac{x^n}{n!} \ ;$$

$$Pe(x) = \sum_{n \geqslant 0} Pe^n \frac{x^{n+1}}{(n+1)!} \ ;$$

$$St(x) = \sum_{n \geqslant 0} St^n \frac{x^n}{n!} \ ;$$

$$As(x) = \sum_{n \geqslant 0} As^n x^{n+2} \ ;$$

$$Cy(x) = \sum_{n \geqslant 0} Cy^n \frac{x^{n+1}}{n+1} \ .$$

Lemma. The following relations hold:

$$d\Delta(x) = x\Delta(x) \ ;$$

$$dI(x) = 2xI(x) \ ;$$

$$dPe(x) = Pe^2(x) \ ;$$

$$dSt(x) = \big(x + Pe(x)\big)St(x) \ ;$$

$$dAs(x) = As(x)\frac{d}{dx}As(x) \ ;$$

$$dCy(x) = As(x)\frac{d}{dx}Cy(x) \ .$$

Theorem. Let $F\colon \mathscr{P} \to \mathscr{P}[t] \;:\; P \mapsto F(P;t)$ be a *linear* map such that

$$F(dP^n;t) = \frac{\partial}{\partial t} F(P^n;t) \;\text{ and }\; F(P^n;0) = P^n$$

for any polytope P^n. Then

$$F(P^n;t) = \sum_{k=0}^{n} d^k P^n \frac{t^k}{k!}.$$

Let $P(x) = \sum_{n\geqslant 0} \lambda_n P^n x^n \in \mathscr{P} \otimes \mathbb{Q}[[x]]$ be a generating series of a family $\{P^n\}$ of polytopes. Set

$$P(t,x) = \sum_{n\geqslant 0} \lambda_n F(P^n;t) x^n.$$

We have $P(0,x) = P(x)$.

Thus, for the series

$$\Delta(x),\; I(x),\; Pe(x),\; St(x),\; As(x),\; Cy(x)$$

we obtain the series

$$\Delta(t,x),\; I(t,x),\; Pe(t,x),\; St(t,x),\; As(t,x),\; Cy(t,x).$$

Theorem.

$$\frac{\partial}{\partial t}\Delta(t,x) = x\Delta(t,x)\ ;$$

$$\frac{\partial}{\partial t}I(t,x) = 2xI(t,x)\ ;$$

$$\frac{\partial}{\partial t}Pe(t,x) = Pe^2(t,x)\ ;$$

$$\frac{\partial}{\partial t}St(t,x) = (x+Pe(t,x))St(t,x)\ ;$$

$$\frac{\partial}{\partial t}As(t,x) = As(t,x)\frac{\partial}{\partial x}As(t,x)\ ;$$

$$\frac{\partial}{\partial t}Cy(t,x) = As(t,x)\frac{\partial}{\partial x}Cy(t,x)\ .$$

Four of these equations, namely those corresponding to the series Δ, I, Pe and St, are ordinary differential equations. Their solutions are completely determined by the initial data $P(0,x) = P(x)$ and are given by explicit formulae

$$\Delta(t,x) = \Delta(x)e^{tx}\ ; \qquad I(t,x) = I(x)e^{2tx}\ ;$$

$$Pe(t,x) = \frac{Pe(x)}{1-tPe(x)}\ ;\ \ St(t,x) = St(x)\frac{e^{tx}}{1-tPe(x)}\ .$$

Quasilinear Burgers–Hopf Equation

The Hopf equation (Eberhard F.Hopf, 1902–1983) is the equation

$$U_t + \varphi(U)U_x = 0.$$

The Hopf equation with $\varphi(U) = U$ is a limit case of the following equations:

$U_t + UU_x = \mu a U_{xx}$ (the Burgers equation),
$U_t + UU_x = \varepsilon a U_{xxx}$ (the Korteweg–de Vries equation).

The Burgers equation (Johannes M.Burgers, 1895–1981) occurs in various areas of applied mathematics (fluid and gas dynamics, acoustics, traffic flow). It used for describing of wave processes with velocity U and viscosity coefficient μ. The case $\mu = 0$ is a prototype of equations whose solution can develop discontinuities (shock waves).

K-d-V equation (Diederik J.Korteweg, 1848–1941 and Hugo M. de Vries, 1848–1935) was introduced as equation for the long waves over water (in 1895). It appears also in plasma physics. Today K-d-V equation is a most famous equation in soliton theory.

Consider the ring homomorphism

$$\xi\colon \mathscr{P} \longrightarrow \mathbb{Z}[\alpha] \;:\; \xi(P^n) = \alpha^n.$$

Then

$$\xi F(P^n; t) = \sum_{k=0}^{n} \xi(d^k P^n)\frac{t^k}{k!} = f(P^n)(\alpha, t)$$

is the face-polynomial.

Set $U(t, x; \alpha, As) = \xi\, As(t, x)$.

Theorem. The function $U(t, x; \alpha, As)$ is the solution of the Hopf equation

$$\frac{\partial}{\partial t} U = U \frac{\partial}{\partial x} U$$

with the initial condition $U(0, x) = \frac{x^2}{1-\alpha x}$.

Corollary. The function $U(t, x; \alpha, As)$ satisfies the equation

$$t(\alpha + t)U^2 - (1 - (\alpha + 2t)x)U + x^2 = 0.$$

Let us consider the Burgers equation

$$\frac{\partial}{\partial t} U = U \frac{\partial}{\partial x} U - \mu a \frac{\partial^2}{\partial x^2} U.$$

Set $U = \sum_{k \geqslant 0} \mu^k U_k$. Then

$$\sum_{k\geqslant 0} \mu^k \Big(\frac{\partial}{\partial t} U_k\Big) = \Big(\sum_{k\geqslant 0} \mu^k U_k\Big)\Big(\sum_{k\geqslant 0} \mu^k \frac{\partial}{\partial x} U_k\Big) - \mu a \sum_{k\geqslant 0} \mu^k \frac{\partial^2}{\partial x^2} U_k.$$

Thus we obtain:

$$\frac{\partial}{\partial t} U_0 = U_0 \frac{\partial}{\partial x} U_0, \qquad \frac{\partial}{\partial t} U_1 = \frac{\partial}{\partial x} (U_0 U_1) - a \frac{\partial^2}{\partial x^2} U_0.$$

Lemma. The general solution to the equation

$$\frac{\partial}{\partial t} V = \frac{\partial}{\partial x} (UV) - a \frac{\partial^2}{\partial x^2} U \quad \text{with} \quad V(0,x) = \psi(x)$$

have the form $V = V_0 + V_1$ where

$$\frac{\partial}{\partial t} V_0 = \frac{\partial}{\partial x} (UV_0) - a \frac{\partial^2}{\partial x^2} U \quad \text{with} \quad V_0(0,x) = 0,$$

$$\frac{\partial}{\partial t} V_1 = \frac{\partial}{\partial x} (UV_1) \quad \text{with} \quad V_1(0,x) = \psi(x).$$

Set

$$U = \sum_{l \geqslant 0} b_l(x) \frac{t^l}{l!}, \qquad V_0 = \sum_{k \geqslant 1} c_k(x) \frac{t^k}{k!}.$$

Then we obtain

$$c_1(x) = -ab_0''(x),$$

$$c_n(x) = \frac{\partial}{\partial x}\left(\sum_{l=1}^{n-1} \binom{n-1}{l} b_l(x) c_{n-1-l}(x)\right) - ab_{n-1}''(x), \ n > 1.$$

Set $V(t, x; \alpha, Cy) = \int\limits_0^x \xi Cy(t, x) dx$.

Theorem. The function $V(t, x; \alpha, Cy)$ is the solution of the equation

$$\frac{\partial}{\partial t} V = \frac{\partial}{\partial x}(UV) \quad \text{with} \quad V(0, x) = -\frac{1}{\alpha} \ln(1 - \alpha x),$$

where U is the solution of the Hopf equation

$$\frac{\partial}{\partial t} U = U \frac{\partial}{\partial x} U \quad \text{with} \quad U(0, x) = \frac{x^2}{1 - \alpha x}.$$

Part II

Abstract

We construct a homomorphism from the ring of convex polytopes to the ring of quasisymmetric functions over integers. Two polytopes have the same image if and only if their flag-vectors coincide.

We describe the image of this homomorphism in terms of functional equations, which are perfected form of the Bayer-Billera relations (generalized Dehn-Sommerville relations).

Quasisymmetric functions form a ring containing the ring of classical symmetric functions. They are indexed by compositions of positive integers in the way similar to how symmetric functions are indexed by partitions. Quasisymmetric functions arise naturally in diverse areas of mathematics such as combinatorics, noncommutative geometry, algebraic topology, Hecke algebras and quantum groups.

We construct a homomorphism from the ring of convex polytopes to the ring of quasisymmetric functions over integers. Two polytopes have the same image if and only if their flag-vectors coincide.

We show that the image over the rational numbers of this homomorphism is a free commutative polynomial algebra and describe this image over the integers in terms of functional equations.

Contents

Flag f-vectors

For an n-dim polytope P^n the faces of all dimensions i, $0 \leqslant i \leqslant n-1$, form a partially ordered set called a *face poset* $fp(P^n)$.

Let $\omega = (i_1 < \ldots < i_k)$, where $i_1 \geqslant 0$ and $i_k \leqslant n-1$. Define $f_\omega(P^n)$ as the *number* of all chains $\{P^{i_1} \subset \ldots \subset P^{i_k}\}$ in $fp(P^n)$.

Definition. $\operatorname{flag}(P^n) = (f_\omega : \omega \subseteq [0, n-1])$, where $f_\varnothing = 1$.

Theorem. (M.Bayer, L.Billera, 1985)
For n-dim polytopes

$$\dim \operatorname{aff}\{\operatorname{flag}(P^n)\} = c_n - 1,\ n \geqslant 1,$$

where c_n is the n-th Fibonacci number.

Note: For *simple* n-dim polytopes

$$\dim \operatorname{aff}\{\operatorname{flag}(P^n)\} = [\tfrac{n}{2}].$$

By definition $c_n = c_{n-1} + c_{n-2},\ n > 1,\ c_0 = 1,\ c_1 = 1.$

The first Fibonacci numbers are:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21$$

Faces-operator

Let P^n be a polytope. Denote by $d_k P^n,\ k \geqslant 0$, the *disjoint union* of all its $(n-k)$-dimensional faces.

Lemma. There is a linear operator of degree $-2k$

$$d_k\colon \mathscr{P} \longrightarrow \mathscr{P}$$

such that

$$d_k P_1^{n_1} P_2^{n_2} = \sum_{i+j=k} (d_i P_1^{n_1})(d_j P_2^{n_2}).$$

Definition. The faces-operator is the linear map

$$\Phi(t)\colon \mathscr{P} \longrightarrow \mathscr{P}[t] \ :\ \Phi(t)(P^n) = \sum_{k=0}^{\infty} d_k P^n t^k.$$

Theorem.

1. $\Phi(t)$ is a ring homomorphism.
2. $\Phi(t)(P^n) = e^{td}(P^n)$ if and only if P^n is simple.
3. The composition

$$\Phi(\alpha, t)\colon \mathscr{P} \xrightarrow{\Phi(t)} \mathscr{P}[t] \xrightarrow{\xi(\alpha)} \mathbb{Z}[\alpha, t],$$

where $\xi(\alpha)(P^n) = \alpha^n$ and $\xi(\alpha)t = t$, is the face-polynomial ring homomorphism f.

Flag-vector polynomial

Let $\Phi(t_1)$ be a faces-operator.

Consider the extension of the faces-operator $\Phi(t_m)$

$$\widehat{\Phi}(t_m)\colon \mathscr{P}[t_1,\dots,t_{m-1}] \longrightarrow \mathscr{P}[t_1,\dots,t_m],\ m>1,$$

such that $\widehat{\Phi}(t_m)(t_i) = t_i,\ 1 \leqslant i < m$.

Introduce the ring homomorphisms

$$\mathscr{F}(t_1,\dots,t_m)\colon \mathscr{P} \longrightarrow \mathscr{P}[t_1,\dots,t_m],\ m>1,$$

by induction as the compositions

$$\mathscr{P} \xrightarrow{\mathscr{F}(t_1,\dots,t_{m-1})} \mathscr{P}[t_1,\dots,t_{m-1}] \xrightarrow{\widehat{\Phi}(t_m)} \mathscr{P}[t_1,\dots,t_m].$$

We obtain the operator

$$\mathscr{F}(t_1,\dots,t_m) = 1 + \sum_{q\geqslant 1}\ \sum_{|J|=q} d_J\,\zeta(t^J)$$

where $J=(j_1,\dots,j_k),\ j_i \neq 0,\ i=1,\dots,k,\ 1\leqslant k\leqslant m$, $|J| = j_1+\dots+j_k, \quad d_J = d_{j_k}\cdots d_{j_1}, \quad t^J = t_1^{j_1}\cdots t_k^{j_k}$ and

$$\zeta(t^J) = \sum_{1\leqslant l_1<\dots<l_k\leqslant m} t_{l_1}^{j_1}\cdots t_{l_k}^{j_k}.$$

Application of quasisymmetric functions

Definition. A *composition* J of a number n is an ordered set $J = (j_1, \dots, j_k)$, $j_i \geqslant 1$, such that $n = j_1 + j_2 + \dots + j_k$. Let us denote $|J| = n$.

The number of compositions of n into exactly k parts is given by the binomial coefficient $\binom{n-1}{k-1}$.

Definition. A *quasisymmetric monomial* in m variables for a composition J is the polynomial

$$\zeta(t^J) = \sum_{1 \leqslant l_1 < \dots < l_k \leqslant m} t_{l_1}^{j_1} \dots t_{l_k}^{j_k}$$

Lemma. The polynomial $f \in \mathbb{Z}[t_1, \dots, t_m]$ is a linear combination of quasisymmetric monomials if and only if $f(t_1, \dots, t_m)$ satisfies the following conditions:

$$f(0, t_1, t_2, \dots, t_{m-1}) = f(t_1, 0, t_2, \dots, t_{m-1}) =$$
$$= f(t_1, t_2, 0, \dots, t_{m-1}) = \dots = f(t_1, t_2, t_3, \dots, t_{m-1}, 0).$$

Let $\mathrm{QSym}^{2n}(m) \subset \mathbb{Z}[t_1, \ldots, t_m]$ be the subgroup generated by the quasisymmetric monomials $\zeta(t^J)$ corresponding to all compositions $J = (j_1, \ldots, j_k)$ of n, where $k \leqslant m$. It is easy to see that for $k \leqslant m-1$

$$\zeta(t^J)(t_1, \ldots, t_{m-1}, 0) = \zeta(t^J)(t_1, \ldots, t_{m-1}).$$

Set $\mathrm{QSym}^{2n} = \varprojlim_m \mathrm{QSym}^{2n}(m)$.

Lemma. $\mathrm{QSym} = \sum_{n \geqslant 0} \mathrm{QSym}^{2n}$ is a graded subring in

$$V = \sum_{n \geqslant 0} V^{2n} = \varprojlim_m \mathbb{Z}[t_1, \ldots, t_m],$$

where $\deg t_k = 2$.

Theorem. (M.Hazewinkel, 2001)
The algebra of quasisymmetric functions QSym is a free commutative algebra of polynomials over the integers.

Since $\dim \mathrm{QSym}^{2n} = 2^{n-1}$, $n \geqslant 1$, the numbers β_i of the multiplicative generators of degree $2i$ of QSym can be found by a recursive relation:

$$\frac{1-t}{1-2t} = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^{\beta_i}}$$

Denote by $\mathscr{F}(\alpha;\boldsymbol{t})$ the ring homomorphism

$$\mathscr{P} \xrightarrow{\mathscr{F}(\boldsymbol{t})} \mathscr{P} \otimes \mathrm{QSym} \xrightarrow{\widehat{\varepsilon}(\alpha)} \mathrm{QSym}[\alpha] \subset \mathbb{Z}[\alpha;\boldsymbol{t}],$$

where $\widehat{\varepsilon}(\alpha)$ is the extension of the ring homomorphism

$$\varepsilon(\alpha)\colon \mathscr{P} \longrightarrow \mathbb{Z}[\alpha] \,:\, \varepsilon(\alpha)(P^n) = \alpha^n,\ n \geqslant 0,$$

such that $\widehat{\varepsilon}(\alpha)(t_i) = t_i$.

Lemma. Let P^n be an n-dim polytope. Then

$$\mathscr{F}(P^n)(\alpha;\boldsymbol{t}) = \alpha^n + \sum_{q=1}^{n} \alpha^{n-q} \sum_{|J|=q} f_{\omega(J)}(P^n)\zeta(\boldsymbol{t}^J)$$

is a homogeneous polynomial of degree $2n$.

Here $f_{\omega(J)}(P^n)$ for $J=(j_1,\ldots,j_k)$ is the ω-flag number of P^n with $\omega=\omega(J)=(i_1<\cdots<i_k)$, where $i_1=n-q,\ \ldots,\ i_l=i_{l-1}+j_{k-l+2},\ \ldots,\ i_k=i_{k-1}+j_2$ and $q=|J|$.

Theorem. The image of the ring homomorphism

$$\mathscr{F}(\alpha, \boldsymbol{t})\colon \mathscr{P}^{2n} \longrightarrow \mathrm{QSym}(m)[\alpha],\ m \geqslant n,$$

consists of all homogeneous polynomials $f(\alpha, t_1, \ldots, t_m)$ of degree n satisfying the equations:

1. $$f(\alpha, t_1, -t_1, t_3, \ldots, t_m) = f(\alpha, 0, 0, t_3, \ldots, t_m);$$
$$f(\alpha, t_1, t_2, -t_2, t_4, \ldots, t_m) = f(\alpha, t_1, 0, 0, t_4 \ldots, t_m);$$
$$\ldots$$
$$f(\alpha, t_1, \ldots, t_{m-2}, t_{m-1}, -t_{m-1}) = f(\alpha, t_1, \ldots, t_{m-2}, 0, 0);$$

2. $$f(-\alpha, t_1, \ldots, t_{m-1}, \alpha) = f(\alpha, t_1, \ldots, t_{m-1}, 0);$$

These equations are a perfected form of the Bayer-Billera (generalized Dehn-Semmerville) relations.

Corollary. The image of the restriction of $\mathscr{F}(\alpha, \boldsymbol{t})$ on $\mathscr{P}_S^{2n}$ consists of all homogeneous polynomials

$$f(\alpha, t_1, \ldots, t_m) = f_1(\alpha, t_1 + \ldots + t_m)$$

where $f_1(\alpha, t)$ is a homogeneous polynomial in two variables of degree n satisfying the equations

$$f_1(-\alpha, \alpha + t) = f_1(\alpha, t).$$

This equation is a perfected form of the classical Dehn-Sommerville relations (see slide 13).

Theorem. The image of the ring homomorphism

$$\mathscr{F}(\alpha, \boldsymbol{t})\colon \mathscr{P} \otimes \mathbb{Q} \longrightarrow \mathrm{QSym}(m)[\alpha] \otimes \mathbb{Q}$$

is a free polynomial algebra with the structure of the graded Hopf algebra dual to the free associative Lie-Hopf algebra $\mathbb{Q}\langle u_1, u_2\rangle$, where $\deg u_i = 2i$ and

$$\Delta u_i = u_i \otimes 1 + 1 \otimes u_i, \quad i = 1, 2.$$

Since dimension of the $2n$-th graded component of the ring $\mathscr{F}(\alpha, \boldsymbol{t})(\mathscr{P} \otimes \mathbb{Q})$ is equal to the n-th Fibonacci number c_n, there is a representation of the generating series of Fibonacci numbers as an infinite product:

$$\frac{1}{1 - t - t^2} = \sum_{n=0}^{\infty} c_n t^n = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^{k_i}},$$

where k_i is the number of multiplicative generators of degree $2i$ in the polynomial ring $\mathscr{F}(\alpha, \boldsymbol{t})(\mathscr{P} \otimes \mathbb{Q})$. The infinite product converges absolutely in the interval $|t| < \frac{\sqrt{5}-1}{2}$. The numbers k_n satisfy the inequalities $k_{n+1} \geqslant k_n \geqslant N_n - 2$, where N_n is the number of the decompositions of n into the sum of odd numbers.

References

[1] A. Baker, B. Richter, *Quasisymmetric functions from a topological point of view.*, Math. Scand. 103, 2008, 208-242.

[2] M. M. Bayer, L. J. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets.*, Invent. Math. 79, 1985, 143-157.

[3] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics.*, AMS, University Lecture Series, v. 24, Providence,RI, 2002.

[4] V. M. Buchstaber, E. V. Koritskaya, *The Quasi-Linear Burgers-Hopf Equation and the Stasheff Polytopes.*, Funct. Anal. Appl., 41:3, 2007, 196–207.

[5] V. M. Buchstaber, *f-polynomials of simple polytopes and two-parameter Todd genus.*, Russian Math. Surveys, v. 63, Issue 3, 2008, 554–556.

[6] V. M. Buchstaber, *Ring of Simple Polytopes and Differential Equations.*, Proceedings of the Steklov Institute of Mathematics, v. 263, 2008, 1–25.

[7] K. T. Chen, R. H. Fox, R. C. Lyndon, *Free differential calculus*, IV, Ann. Math. 68, 1958, 81-95.

[8] E.-M. Feichtner, B. Sturmfels, *Matroid polytopes, nested sets and Bergman fans*, Portugaliae Mathematica 62(2005), 437-468; arXiv: math/0411260 v1[math.CO]

[9] M. Hazewinkel, *The Algebra of Quasisymmetric Functions is free over integers.* Advances in Mathematics, 164, 2001, 283-300.

[10] V. Klee, *A combinatorial analogue of Poincaré's duality theorem.*, Canad. J. Math., 16, 1964, 517–531.

[11] A. Postnikov, *Permutohedra, associahedra, and beyond.*, arXiv: math/0507163 v1[math.CO], 7 Jul 2005.

[12] A. Postnikov, V. Reiner, L. Williams, *Faces of generalized permutohedra.*, arXiv: math/0609184 v2 [math.CO] 18 May 2007.

[13] R. P. Stanley, *Flag f-vectors and the cd-index.*, Math. Z. 216, 1994, 483–499.

[14] A. Zelevinsky, *Nested complexes and their polyhedral realizations.*, arXiv: Math/0507277, v.4 [math.CO] 13 May 2006.

[15] Günter M. Ziegler, *Face numbers of 4-polytopes and 3-spheres.*, Proceedings of the ICM 2002 Beijing, Volume III, 2004, 625–634; arXiv: math/0208073 v.2 [math.MG] 20 Jun 2003.