Geometric aspects of polynomial interpolation in more variables

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Univariate polynomial interpolation

A polynomial of degree at most d over a field $\mathbb{K}$

$$f(x) = a_0 + a_1 x + \ldots + a_d x^d \in \mathbb{K}[x]$$

depends on $d+1$ parameters: its coefficients.

Fix $d+1$ distinct points

$$x_0, \ldots, x_d \in \mathbb{A}^1_{\mathbb{K}} \cong \mathbb{K}$$

and set the values

$$f(x_i) = f_i \in \mathbb{K}, \quad i = 0, \ldots, d.$$

Then there is a unique such polynomial $f(x)$.

Reason: linear algebra plus the fact that there is no non–zero polynomial of degree d with zeros at $x_0, \ldots, x_d$.

Univariate Hermite interpolation

Fix distinct points and positive integers

$$x_1, \ldots, x_h \in \mathbb{A}^1_{\mathbb{K}}, \quad m_1, \ldots, m_h \in \mathbb{N}, \quad m_1 + \ldots + m_h = d + 1$$

and set the values of the derivatives

$$f^{(j-1)}(x_i) = f_{i,j}, \quad i = 1, \ldots, h, \quad j = 1, \ldots, m_i$$

Again there is a unique such polynomial $f(x)$, because there is no non–zero polynomial of degree d with zeros of multiplicities at least $m_1, \ldots, m_h$ at $x_1, \ldots, x_h$, i.e. Ruffini's theorem holds.

Consequence: given a differentiable function $F(x)$ of a real variable, we can uniquely approximate it with a polynomial of degree d by fixing $d + 1$ values of $F(x)$ and of its derivatives.

What is the situation for $n \geq 2$ variables?

The multivariate case

In general a polynomial $f(x_0, ..., x_n) \in \mathbb{K}[x_0, ..., x_n]$ of degree at most d depends on

$$\boxed{N_{n,d} + 1 := \binom{d+n}{n}}$$

parameters, i.e. its coefficients. Fix points, positive integers and constants in $\mathbb{K}$

$$p_i = (x_{i,1}, ..., x_{i,n}) \in \mathbb{A}^n_{\mathbb{K}} \cong \mathbb{K}^n, \quad m_i > 0, \quad f_{i,j} \in \mathbb{K}, \quad i = 1, ..., h, \quad j = 1, ..., m_i$$

with the condition

$$\boxed{\sum_{i=1}^{h} \binom{m_i + n - 1}{n} = N_{n,d} + 1}$$

and impose

$$D^{(j-1)} f(p_i) = f_{i,j}, \quad i = 1, ..., h, \quad j = 1, ..., m_i$$

where $D^{(k)}$ is any derivative of order k. Is the resulting polynomial f uniquely determined?

This is a linear system in the coefficients of f, whose associated homogeneous system is

$$D^{(j-1)} f(p_i) = 0, \quad i = 1, ..., h, \quad j = 1, ..., m_i$$

Is the only solution to this system the 0 polynomial?

It is convenient to address this question in a more general, different, geometric setting.

Linear systems with multiple base points

- X is a projective, complex manifold of dimension n.
- $\mathcal{L}$ is a linear system of codimension one subvarieties, i.e. divisors, on X.
- $p_1, \ldots, p_h$ are distinct points on X.
- $m_1, \ldots, m_h$ are positive integers.
- $\mathcal{L}(-\sum_{i=1}^h m_i p_i) \subseteq \mathcal{L}$ is the sublinear system formed by all divisors in $\mathcal{L}$ having multiplicity at least m_i at the base points p_i, $i = 1, \ldots, h$, i.e.

  the local equation of the divisors in $\mathcal{L}(-\sum_{i=1}^h m_i p_i)$ vanishes at p_i with all its derivatives of order $\ell \leq m_i - 1$.

  This imposes

$$\boxed{\sum_{i=1}^{h} \binom{m_i + n - 1}{n}}$$

  linear conditions on $\mathcal{L}$.

The expected dimension of $\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)$ is:

$$e := \max\{\dim(\mathcal{L}) - \sum_{i=1}^{h} \binom{m_i + n - 1}{n}, -1\}$$

By linear algebra

$$\dim(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)) \geq \mathrm{expdim}(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i))$$

$\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)$ is said to be non–special if

$$\dim(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)) = \mathrm{expdim}(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i))$$

it is called special otherwise: in this case the conditions imposed on $\mathcal{L}$ are dependent.

Note: according to the definition, an empty system is non–special.

The main question in this talk will be the ...

Problem (The dimensionality problem)

Classify all special linear systems.

Though more refined questions could be asked.

However even this question is far too complicated! The answer depends on too many circumstances, e.g., it depends on the position of the points $p_1, \ldots, p_h$ on X.

Example (An easy example of special position of the points)

If $X = \mathbb{P}^r$, and $p_1, \ldots, p_h$ are on a line, they give dependent conditions to all hypersurfaces of degree $d \leq h - 2$.

In any case, $\dim(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i))$ is upper–semicontinuous in the position of the points $p_1, ..., p_h$, hence it reaches its minimum for $p_1, ..., p_h$ in sufficiently general position on X, i.e. for $(p_1, ..., p_h)$ in a suitable not empty Zariski open subset $U_{m_1,\ldots,m_h}$ of X^h.

The general dimensionality problem

Take $p_1, \ldots, p_h$ sufficiently general on X and set

$$\mathcal{L}(-\sum_{i=1}^{h} m_i p_i) := \mathcal{L}(m_1, \ldots, m_h) = \mathcal{L}(m_1^{l_1}, \ldots, m_t^{l_t})$$

The case $t = 1$ is called homogeneous. Define the general dimension of the system as

$$\mathrm{gendim}(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)) := \dim(\mathcal{L}(m_1, \ldots, m_h))$$

Problem (The GDP)

If $p_1, \ldots, p_h$ are general, is $\mathrm{gendim}(\mathcal{L}(-\sum_{i=1}^{h} m_i p_i))$ equal to the expected dimension of $\mathcal{L}(-\sum_{i=1}^{h} m_i p_i)$? If not then classify all systems $\mathcal{L}(m_1, \ldots, m_h)$ which are special.

The GDP is easy in the curve case (the answer is that no system with general base points is special in this case), but very complicated in general as soon as $n = \dim(X) \geq 2$.

However a trivial situation is when $m_i = 1$: general simple base points always impose independent conditions.

A wise reduction

Given the complexity of the problem, it is wise to consider particular varieties X and linear systems $\mathcal{L}$ on them. Typically we take

$$X = \mathbb{P}^n, \qquad \mathcal{L} = \mathcal{L}_{n,d} := \text{all degree} \quad d \quad \text{hypersurfaces.}$$

In this case

$$e := \mathrm{expdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)) = \max\{\mathrm{virtdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)), -1\}$$

where

$$\boxed{v := \mathrm{virtdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)) = \binom{d+n}{n} - 1 - \sum_{i=1}^{h} \binom{m_i + n - 1}{n}}$$

is the so-called virtual dimension of the system.

The GDP here coincides with the original polynomial interpolation problem. This is in general widely open, and there is even no conjectural answer to it.

Except for the planar case ...

The planar case

- $p_1, \ldots, p_h$ general points in $\mathbb{P}^2$, $m_1, \ldots, m_h \in \mathbb{N}$ multiplicities

$$\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_h) = \mathcal{L}_d(m_1^{\ell_1}, \ldots, m_t^{\ell_t})$$

  is the linear system of plane curves of degree $d > 0$ having multiplicity at least m_i at p_i for each $i = 1, \ldots, h$.

- The *virtual dimension* of $\mathcal{L}$ is

$$\boxed{v := v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2}$$

- The *expected dimension* is

$$\boxed{e := e(\mathcal{L}) = \max\{-1, v\}}$$

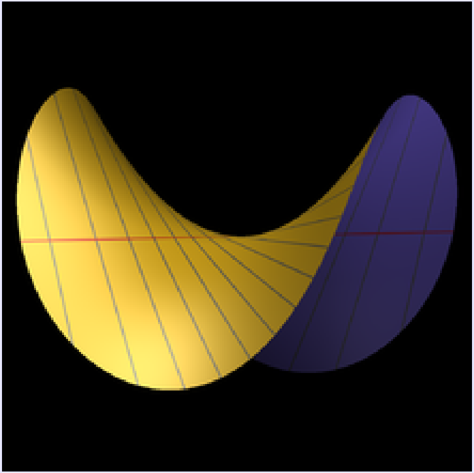
- $\mathcal{L}$ is *special* if

$$\boxed{\dim(\mathcal{L}) > e(\mathcal{L})}$$

Blow–up

One may formulate this on the blow–up of $\mathbb{P}^2$ at $p_1, \ldots, p_h$.

The blow–up is an algebraic surgery operation which substitutes to a point p in the plane (or on a surface) an exceptional curve $E \cong \mathbb{P}^1$, with normal bundle of degree -1, called a (-1)–curve. This is shown in red in the picture below:



Line bundles and cohomology

- X is the blow–up of $\mathbb{P}^2$ at $p_1, \ldots, p_h$, with the following divisor classes generating the Picard group $\mathrm{Pic}(X)$, i.e. the group of divisors modulo linear equivalence (which can be identified with the group of line bundles, i.e. vector bundles of rank 1, modulo isomorphism, on X):
  - H the pull–back of a line
  - $E_1, \ldots, E_h$ the exceptional divisors over $p_1, \ldots, p_h$
- The relevant line bundle on X is

$$\mathcal{L} = \mathcal{O}_X(dH - \sum_{i=1}^{h} m_i E_i)$$

  and the virtual dimension is

$$v = \chi(\mathcal{L}) - 1 = h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) - 1.$$

- So $\mathcal{L}$ is non special if and only if

$$\boxed{h^0(X, \mathcal{L}) \cdot h^1(X, \mathcal{L}) = 0}$$

  i.e. $\mathcal{L}$ has natural cohomology.

The Segre–Harbourne–Gimigliano–Hirschowitz conjecture

- Naive conjecture: for general base points, $\mathcal{L}$ always has the expected dimension.
- This is wrong: e.g. look at $\mathcal{L}_2(2^2)$ or to $\mathcal{L}_4(2^5)$.
- (-1)–special systems: if $\mathcal{L}$ is not empty and C is a (-1)–curve on X then

$$\boxed{\mathcal{L} \cdot C = -N < 0 \Longrightarrow h^1(X, \mathcal{L}) \geq \binom{N}{2}}$$

  The two above examples are of this type.
- a (-1)–curve on X is a curve $C \cong \mathbb{P}^1$, with $C^2 = -1$, equivalently it can be blown down to a smooth point (Castelnuovo's theorem).

Conjecture (SHGH)

$\mathcal{L}$ is special if and only if it is (-1)–special.

Equivalently, $\mathcal{L}$ is special if and only if the general curve in $\mathcal{L}$ has some multiple component (which turns out to be a fixed (-1)–curve).

Nagata's conjecture

Conjecture (Nagata, 1959)

If $h > 9$, the base points are very general and $\mathcal{L}_d(m_1, \ldots, m_h)$ is not empty, then

$$\boxed{\sum_{i=1}^{h} m_i < d\sqrt{h}}$$

Conjecture (Stronger Nagata's Conjecture (SNC))

If the $h \geq 0$ base points are very general and $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_h)$ is not empty, not (-1)–special, then

$$\boxed{\sum_{i=1}^{h} m_i^2 \leq d}$$

In particular, if $C \in \mathcal{L}$ is an irreducible curve different from a (-1)–curve, one has $C^2 \geq 0$ on the blow–up.

The SHGH Conjecture implies SNC, the converse does not hold. However Nagata's conjecture is an asymptotic form of the SHGH Conjecture.

Nagata's original paper dealt with a negative answer to Hilbert's fourteenth problem.

Nagata's conjecture arises also in other contexts, like in symplectic packing problems (D. MacDuff–L. Polterovich, P. Biran).

Definition (MSC)

Given $p_1, \ldots, p_h \in \mathbb{P}^2$, the Multipoint Seshadri Constant is defined as

$$\epsilon(\mathbb{P}^2, p_1, \ldots, p_h) := \inf\{\frac{d}{\sum_{i=1}^h m_i} : \mathcal{L}_d(-\sum_{i=1}^h m_i p_i) \neq \emptyset, \quad m_i \geq 0, \quad \sum_{i=1}^h m_i > 0\}$$

If the points are very general denote it by ϵ_h.

One has

$$\epsilon(\mathbb{P}^2, p_1, \ldots, p_h) \leq \frac{1}{\sqrt{h}}$$

Nagata's conjecture asserts equality holds for very general points

$$\epsilon_h = \frac{1}{\sqrt{h}}$$

Nagata's conjecture and the Mori cone

If X is a complex, projective manifold, the Mori cone $\overline{NE}(X)$ of X is the closure of the convex cone spanned by the classes of effective curves inside $N_1(X)$, the $\mathbb{R}$–vector space dual to the space of $\mathbb{R}$–divisors modulo numerical equivalence.

If X is the plane blown up at h points and ℓ is the class of a line in $N_1(X) \cong \mathrm{Pic}(X) \otimes \mathbb{R}$, consider the quadratic cone in $N_1(X)$

$$Q := \{\alpha : \alpha^2 \geq 0, \alpha \cdot \ell \geq 0\}$$

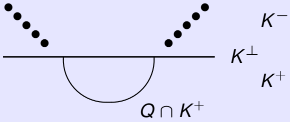
Conjecture (Strong Nagata's Conjecture in Mori's setting)

If X is the plane blown up at h very general points, then

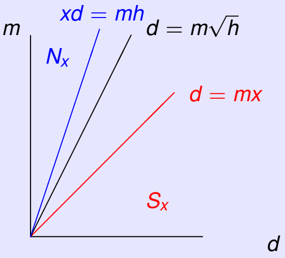
$$\overline{NE}(X) = Q + \sum_i E_i$$

where E_i are the classes of (-1)–curves on X.

The figure illustrates the case $h \geq 10$. The • denote the classes of (-1)–curves. The curved boundary of the Mori cone in the K^+ region has not yet been proved.



For homogenous linear systems Nagata´s and Strong Nagata´s conjectures coincide.



$0 < x \leq \sqrt{h}$

- If for all $(d, m) \in S_x$ the system $\mathcal{L}_d(m^h)$ is non–special, then for all $(d, m) \in N_x$ the system $\mathcal{L}_d(m^h)$ is empty;
- if for all $(d, m) \in N_x$ the system $\mathcal{L}_d(m^h)$ is empty, then for all $(d, m) \in S_x$ the system $\mathcal{L}_{kd}((km)^h)$ is non–special for $k >> 0$, actually S_x is contained in the ample cone.

Results (not comprehensive)

- SHGH holds for $h \leq 9$ (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986).
- SHGH holds for $m_i \leq 11$ (Dumnicki–Jarnicki, 2005; Arbarello–Cornalba, 1981: $m_i = 2$; Hirschowitz, 1985: $m_i \leq 3$; Lorentz–Lorentz, 1986; Mignon, 1998: $m_i \leq 4$; Yang, 2004: $m_i \leq 7$).
- SHGH holds for $\mathcal{L}_d(m^h)$ for $m \leq 42$ (Dumnicki, 2005; Ciliberto–Miranda, 1998: $m \leq 12$; Ciliberto–Cioffi–Miranda–Orecchia, 2003: $m \leq 20$).
- SHGH holds for $\mathcal{L}_d(m^h)$ for $h = k^2$ points (Evain, 2005; Ciliberto–Miranda, 2006; Roé, 2006; Nagata, 1960 proved Nagata Conjecture in this case).

Hirschowitz and his followers (Gimigliano, Mignon, Evain, etc.) use a degeneration technique called the Horace method (i.e. divide et impera), consisting in exploiting subsequent specializations of the points on curves of (relatively) low degree.

Ciliberto–Miranda's approach is based on a different degeneration technique called the blow–up and twist method, consisting in degenerating the plane together with the linear system.

The first interesting case $h = 10$

The virtual dimension of $\mathcal{L}_d(m^{10})$ is equal to -1, and one expects no such curves, for (d, m) in the following table:

d	m	empty
3	1	easy: cubic through ten general points
19	6	posed by Dixmier, solved by Hirschowitz early 80s
38	12	Gimigliano's thesis
174	55	le cas inviolé, according to A. Hirschowitz, see theorem below
778	246	?
1499	474	?
6663	2107	?
⋮	⋮	?

Theorem (Ciliberto–Miranda, 2008)

$\mathcal{L}_d(m^{10})$ has the expected dimension if $\frac{d}{m} \geq \frac{174}{55}$. In particular $\mathcal{L}_{174}(55^{10})$ is empty.

Theorem (Eckl, Ciliberto—Dumitrescu–Miranda—Roé, 2008)

$$\frac{1}{\sqrt{10}} \sim 0.31622\ldots \geq \epsilon_{10} \geq \frac{117}{370} \sim 0.31621\ldots$$

- To give a pair $(X, \mathcal{L})$, where:
  - X is a projective, n–dimensional toric variety
  - an embedding $X \subset \mathbb{P}^r$ given by the sections of a line bundle $\mathcal{L}$

  is equivalent to the datum of:

  - an n dimensional integral compact convex polytope $P \subset \mathbb{R}^n_+$, determined up to integral affine isomorphisms.

- If

$$P \cap \mathbb{Z}^n = \{m_i = (m_{i1}, \ldots, m_{in}), \quad 0 \leq i \leq r\}$$

  consider the monomial map:

$$\phi_P : x \in (\mathbb{C}^*)^n \to (x^{m_0} : \ldots : x^{m_r}) \in \mathbb{P}^r$$

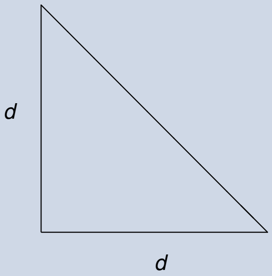
$$\text{where} \quad x = (x_1, \ldots, x_n) \quad \text{and} \quad x^{m_i} = x_1^{m_{i1}} \cdot \ldots \cdot x_n^{m_{in}}$$

- The closure X_P of the image of ϕ_P is the image of X via the map determined by the sections of $\mathcal{L}$.

Example

The *d–Veronese surface* $V_{2,d}$ in $\mathbb{P}^{d(d+3)/2}$ corresponds to the triangle:

$$\Delta_d = \{(x, y) : \quad x \geq 0, y \geq 0, x + y \leq d\}$$



It is the image of the plane via all monomials of degree d.

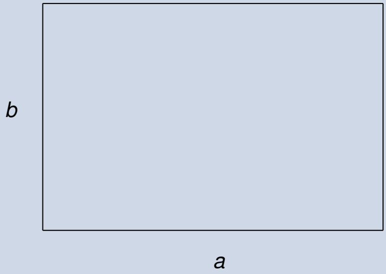
For $d = 1$ we get the plane $\mathbb{P}^2$ itself.

For $d = 2$ we get the famous Veronese surface $V_{2,2}$ of degree 4 in $\mathbb{P}^5$.

Example

$\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{ab+a+b}$ via all monomials of bidegree (a, b) in the variables (x_0, x_1) and (y_0, y_1), corresponds to the rectangle:

$$R_{a,b} = \{(x, y) : \quad 0 \leq x \leq a, \quad 0 \leq y \leq b\}$$



For $a = b = 1$ we get four monomials $X_{ij} = x_i y_j$, $1 \leq i \leq j \leq 2$, verifying a unique quadratic relation

$$\boxed{X_{11}X_{22} - X_{12}X_{21} = 0}$$

i.e. we get a smooth quadric surface in $\mathbb{P}^3$.

Toric degenerations (I)

- Consider a *subdivision* $\mathcal{D}$ of the polytope P defining the toric variety X_P of dimension n, i.e., a finite family of n dimensional convex polytopes whose union is P and any two of them intersect only along a common face.

- $\mathcal{D}$ is called *regular* if there is a piecewise linear, positive function F defined on P such that:

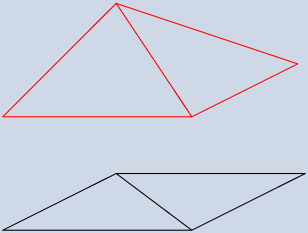
(i) the polytopes of $\mathcal{D}$ are the orthogonal projections on the hyperplane $z = 0$ of $\mathbb{R}^{n+1}$ of the n–dimensional faces of the *graph polytope*

$$G(F) := \{(x, z) \in P \times \mathbb{R} : 0 \leq z \leq F(x)\}$$

which are neither vertical, nor equal to P;

(ii) the function F is *strictly convex* i.e., the hyperplanes determined by each face of $G(F)$ intersect $G(F)$ only along that face.

Example (A simple example of a regular subdivision)

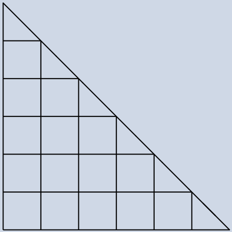


Toric degenerations (II)

- Given a regular subdivison $\mathcal{D}$, there is a one parameter, flat degeneration of X_P, to a reducible toric variety
$$X_0 = \bigcup_{Q \in \mathcal{D}} X_Q$$
- If Q and Q' have a common face R, then
$$X_Q \cap X_{Q'} = X_R$$
- Description of the degeneration:
$$\phi_{\mathcal{D}} : (x, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to (t^{-F(m_0)} x^{m_0} : \ldots : t^{-F(m_r)} x^{m_r}) \in \mathbb{P}^r$$
- $X_t =$ closure of the image of $\phi_{\mathcal{D}}(*, t)$, for $t \neq 0$, is a copy of X_P.
- X_0 is the limit of X_t when t tends to 0.

Example (The quadric degeneration of $V_{2,d}$)

- The regular subdivision illustrated below for $d = 6$

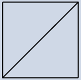
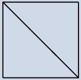


gives a degeneration of $V_{2,d}$ to a union of d planes and $\binom{d}{2}$ quadrics.

- A corresponding strictly convex piecewise linear function is determined by the conditions $F(i,j) = i^2 + j^2$ for i, j non–negative integers.
- The *vertices* of this configuration of planes and quadrics are linearly independent in the ambient $\mathbb{P}^{d(d+3)/2}$ and can be taken as the coordinate points.

Example (Planar degenerations of $V_{2,d}$)

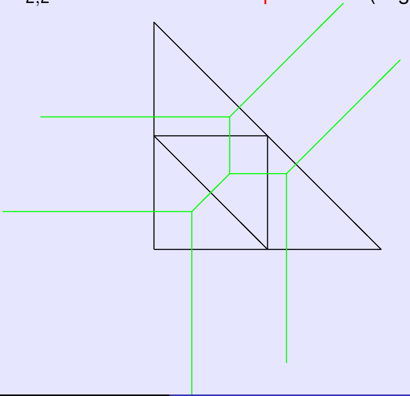
Each quadric can independently degenerate to a union of two planes, in two possible ways:

 or  for each quadric.

This way one finds several different degenerations of the Veronese into a union of planes, i.e. a a planar degeneration.

- There are connections with tropical geometry here: toric degenerations of $V_{2,d}$ are closely related (essentially equivalent, indeed) to plane tropical curves of degree d.

- E.g., we see below a subdivison of Δ_2 corresponding to a planar degeneration of the Veronese surface $V_{2,2}$ in $\mathbb{P}^5$ and a related tropical conic (in green).



Double point interpolation

- This is the only general result in interpolation in more than two variables:

Theorem (Alexander–Hirschowitz, 1996)

$\mathcal{L}_{n,d}(2^h)$ *is non–special unless*

n	*any*	2	3	4	4
d	2	4	4	4	3
h	$2, \ldots, n$	5	9	14	7

- The original, complicated proof used the Horace method. Simplifications, with the same technique, by K. Chandler (2002), Brambilla–Ottaviani (2008).

- The blow–up–and twist method has been applied by E. Postinghel (2010) in her thesis.

- There has been recent activity in trying to find a combinatorial (tropical) proof of this theorem using the above techniques: Draisma (2004), Ciliberto–Dumitrescu–Miranda (2007), for $n = 2$; Brannetti (2007), for $n = 3$.
- Similar ideas also work for other toric varieties (Draisma, 2007).

An idea of the combinatorial proof

- In case $n = 2$, one uses the following:

Lemma (The basic combinatorial lemma)

Suppose there is a planar degeneration D of $V_{2,d}$, and a set of h pairwise disjoint planes of D. Then $\mathcal{L}_d(2^h)$ has the expected dimension

$$\boxed{e = \frac{d(d+3)}{2} - 3h}$$

- **Sketch of the proof of Alexander–Hirschowitz theorem for** $n = 2$: verify that $\mathcal{L}_d(2^h)$ has the expected dimension whenever $d \geq 5$ and

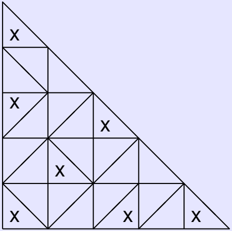
$$h = \lfloor (d+1)(d+2)/6 \rfloor$$

With this number of points, the virtual dimension of $\mathcal{L}_d(2^h)$ is

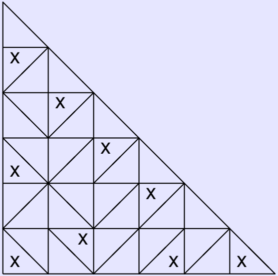
$$v = d(d+3)/2 - 3h = \begin{cases} -1 & \text{if } d \equiv 1, 2 \mod 3 \\ 0 & \text{if } d \equiv 0 \mod 3. \end{cases}$$

Do by hands the cases $5 \leq d \leq 10$, using the basic combinatorial lemma, e.g.

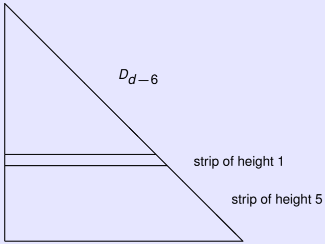
D_5:



D_6:

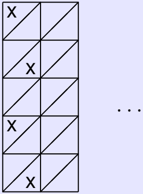


- Then proceed by induction, assuming the result holds for $d-6$.
- Make a planar degeneration of $V_{2,d}$, starting with:

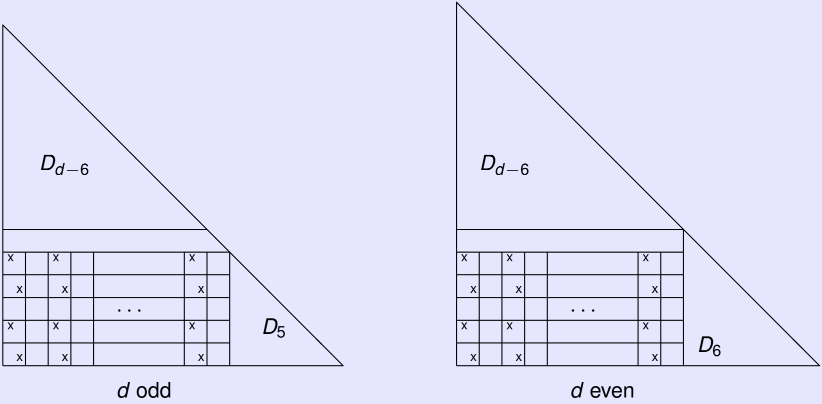


- By induction, we know how to subdivide D_{d-6} in order to get the maximal number of pairwise disjoint triangles.
- Triangulate the central strip of height 1 as you like, and take no triangles there.

- It remains to triangulate the lower strip of height 5 and choose the pairwise disjoint triangles there.
- The strip contains $\lfloor (d+1)/2 \rfloor - 3$ copies of the rectangle:



- On the far right complete using the configurations D_5 and D_6 presented above:



- Brannetti's proof for $r = 3$ is in the same style.

Secant varieties

Why does the basic combinatorial lemma hold? The reason is geometric, related to secant varieties.

- $X \subset \mathbb{P}^r$ a projective variety of dimension n, spanning $\mathbb{P}^r$.

$$S^k(X) = \overline{\bigcup_{p_0,\ldots,p_k \in X, p_0,\ldots,p_k \text{l.i.}} \langle p_0, \ldots, p_k \rangle}$$

  is the *k–secant variety* of X.

- One has

$$\boxed{\text{expdim}(S^k(X)) = \min\{r, (k+1)(n+1) - 1\} \geq \dim(S^k(X))}$$

- X is called k–defective if strict inequality holds.
- By Terracini´s lemma, X is k–defective if and only if $\mathcal{L}(2^{k+1})$ is special, with

$$\mathcal{L} = \text{the linear system of hyperplane sections of } X.$$

Secant varieties are an authentic crossroad in mathematics (and not only!): besides their interest in algebraic geometry, secant varieties arise in a number of other fields, like algebra, representation theory, projective differential geometry, topology, stochastics and algebraic statistics. Recently they became particularly useful in biology, especially in phylogenetics.

- Alexander–Hirschowitz Theorem provides the list of k–defective *Veronese varieties* $V_{n,d} \subset \mathbb{P}^{N_{n,d}}$, $N_{n,d} = \binom{n+d}{n} - 1$.
- This answer the so–called *Waring's problem for forms*.
- Fix positive integers d, k, n. When may we write a form $f(x_0, ..., x_n)$ of degree d as a sum of $k + 1$ d–th powers of linear forms $l_i(x_0, ..., x_n)$, $i = 0, ..., k$, i.e. as

$$f(x_0, ..., x_n) = \sum_{i=0}^{k} l_i(x_0, ..., x_n)^d?$$

- If this happens, we say that the *k–Waring property* holds for f.
- $V_{d,n}$ can be seen as the proportionality classes of non–zero forms of type $l(x_0, ..., x_n)^d$ with $l(x_0, ..., x_n)$ linear. Then the k–Waring property holds for f if and only if $[f] \in S^k(V_{n,d})$.
- The Waring problem has its roots in number theory: given positive integers d, h, may we write any positive integer as a sum of h non-negative d–th powers?
- E.g., for $d = 2$ and $h = 4$, this is affirmatively answered by the celebrated Gauss' Theorem.

- We say that the *strict (n, d, k)–Waring property* holds for a general $[f] \in S^k(V_{n,d})$, if the expression

$$f(x_0, \ldots, x_n) = \sum_{i=0}^{k} l_i(x_0, \ldots, x_n)^d$$

  with l_i linear forms, is unique up to multiplication by a constant. This what biologist call the identifiability condition.

- It is equivalent to the geometric condition: the general point in $S^k(V_{n,d})$ sits in a unique k–dimensional subspace which is $(k+1)$–secant to $V_{n,d}$.

- Strict Waring property provides a *canonical form* for forms enjoying it.

- In general, Waring property gives notions of rank for forms, similar to the rank of tensors which are useful in numerical analysis: f has rank $k+1$ if k is the minimum such that

$$f(x_0, \ldots, x_n) = \sum_{i=0}^{k} l_i(x_0, \ldots, x_n)^d$$

  and has border rank $k+1$ if k is the minimum such that $[f] \in S^k(V_{n,d})$. In general the border rank is smaller than the rank.

Problem

Let D be a planar degeneration of a toric surface X. What is the limit of $S^k(X)$? (Similar questions can be asked for higher dimensional toric varieties.)

- References: Sturmfels—Sullivant (delightful degenerations), Cox–Sidman, Ciliberto–Dumitrescu–Miranda.
- **A remark**: if there is a $(k+1)$–tuple of independent planes in D, then they span a linear space of dimension $3k+2$ sitting in the limit of $S^k(X)$, which is therefore not k–defective. This proves the Basic Combinatorial Lemma.
- **A speculation**: if $S^k(X)$ has the expected dimension $3k+2$, then the limit of $S^k(X)$ is the union of all $(3k+2)$–subspaces spanned by $(k+1)$–tuples of independent planes in D.

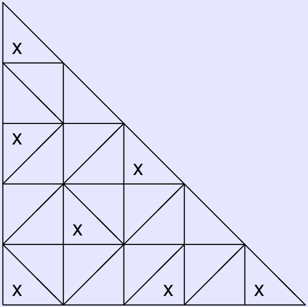
  If this happens, we say the degeneration is almost k–delightful.
- The existence of almost delightful degenerations helps in computing the degree of $S^k(X)$, which is a hard problem, unsolved in general.

Theorem (Hilbert's Theorem; see also Ein–Sheperd-Barron)

There is a unique 6*–space passing through a general point of* $\mathbb{P}^{20}$ *and* 7*–secant the Veronese surface* $V_{2,5}$*, i.e. the strict Waring property holds in this case, giving a canonical form for the general quintic homogeneous polynomial in three variables.*

- Indeed, the configuration of 7 independent planes in D_5 shown below is unique:

D_5:



- There is work in progress on these ideas about: higher multiplicities, higher dimension, influence of the singularities of the degeneration on calculations of the degrees of secant varieties, etc.

Conclusions

- I tried to show how interpolation, originated from elementary analysis and algebra, has deep algebro–geometric aspects as well as applications to other seemingly distant mathematical fields, e.g. symplectic geometry.
- Inside algebraic geometry, we see relations with the projective geometry of secant varieties, which in turn applies again to algebra via tensor rank computation, Waring problem, canonical forms, enumerative problems, etc. They have also recent striking applications to natural science, e.g. to phylogenetics.
- Various techniques are used in this field, among others:
  - degeneration techniques;
  - toric and tropical geometry;
  - combinatorial techniques.
- The field is active and most of the basic, deepest problems like Segre–Harbourne–Gimigliano–Hirschowitz conjecture, Nagata's conjecture, etc. are widely open.