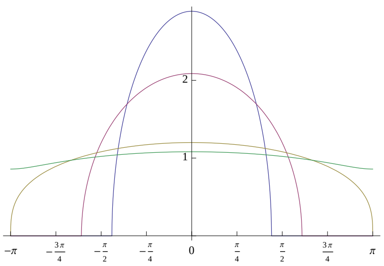
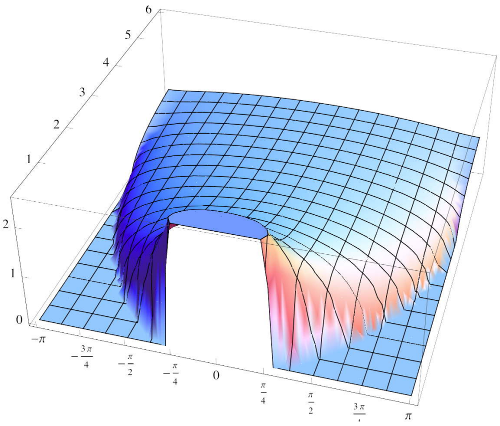
Eigenvalues of large unitary matrices

Thierry Lévy

In this series of lectures, the study of the Brownian motion on unitary groups will be our Ariadne's thread in the exploration of a small piece of mathematical landscape (or is it a maze ?) situated at the interface between probability theory, analysis on Lie groups and combinatorics.

The specific question which we are going to study is the following. Consider the value at an arbitrarily fixed time t of a Brownian motion on the unitary group $\mathrm{U}(N)$ issued from the identity I_N. Informally, this means the following: start from I_N, move around in the unitary group randomly but continuously during a time t, and see where you arrive. The question is: where should we expect to find the eigenvalues $\lambda_1, \ldots, \lambda_N$ of the random unitary matrix which we obtain in this way ? In the limit where N tends to infinity, the answer is that with very high probability, the measure $\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$ is very close to a certain deterministic probability measure ν_t on the set $\mathbb{U}$ of complex numbers of modulus 1. Here is a plot of the density of ν_t with respect to the Lebesgue measure on the circle for $t = 0.1$, $t = 1$, $t = 4$ and $t = 6$ on the left, and for all t between 0.1 and 6 on the right. If you look carefully, you will notice that something special happens at $t = 4$.





There is no simple expression of the density of ν_t, but its moments are known and we shall prove that for all $n \geq 0$,

$$\int_{\mathbb{U}} z^n \, \nu_t(dz) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \binom{n}{k+1} n^{k-1}.$$

The number $\binom{n}{k+1} n^{k-1}$ happens to be the number of k-tuples $(\tau_1, \ldots, \tau_k)$ of transpositions in the symmetric group $\mathfrak{S}_n$ such that the permutation $(1 \ldots n)\tau_1 \ldots \tau_k$ has exactly $k+1$ cycles. This is no coincidence, but an instance of a deep connection between the unitary groups and the symmetric groups, known as the Schur-Weyl duality.

Motivated by this question, the lectures will include an introduction to the following topics: Brownian motion and harmonic analysis on unitary groups, symmetric functions, Schur-Weyl duality, combinatorics of permutations, and free probability. Needless to say, no prior familiarity with any of these subjects is required.

Depending on time and on the wishes of the audience, the end of the series of lectures might be devoted to an application in the context of two-dimensional gauge theories of some of the results which we will have obtained.