What corresponds to broccoli in the real world?

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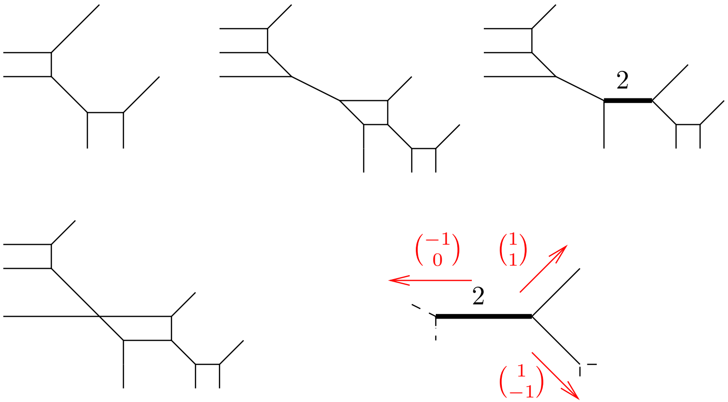
BMS Friday, 2011

Idea of tropical geometry

- Degenerate algebraic curves to tropical curves ("Tropicalization").
- Plane tropical curves are weighted balanced graphs.
- Tropical curves can be studied with combinatorial methods.
- Use tropical geometry to prove theorems about algebraic geometry.

Particularly succesfully applied to questions in enumerative geometry.

Plane tropical curves



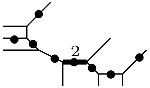
Complex numbers

Definition

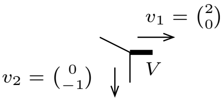
N_d = # nodal rational complex plane curves of degree d through $3d-1$ points (in general position)

N_d^{trop} = # simple rational tropical plane curves of degree d through $3d-1$ points (in general position), counted with complex multiplicity $\text{mult}_{\mathbb{C}}$.

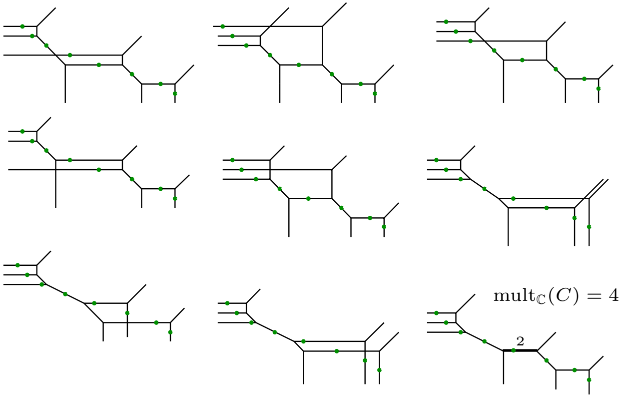
$$\text{mult}_{\mathbb{C}}(C) := \prod_V \text{mult}(V), \quad \text{mult}(V) = |\det(v_1, v_2)|$$



$\text{mult}_{\mathbb{C}}(C) = 4$



Example



$N_3^{\mathrm{trop}} = 12$

Invariance of N_d^{trop}

Example

d	1	2	3	4	5	6
N_d	1	1	12	620	87304	26312976

The numbers N_d are invariant.

Theorem (Mikhalkin's Correspondence Thm, 2005)

$N_d^{\text{trop}} = N_d$.

Invariance of N_d + Correspondence Thm $\Rightarrow$

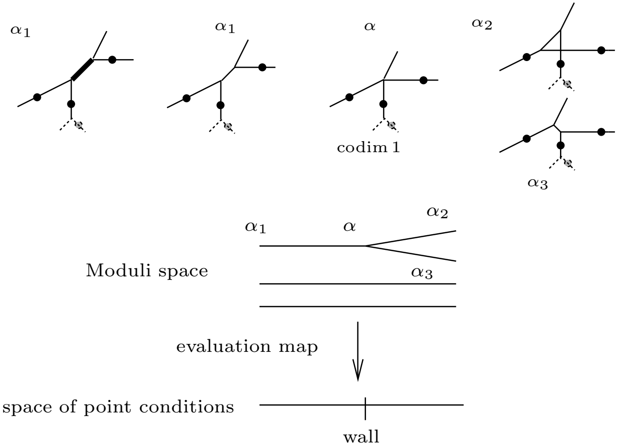
Theorem

N_d^{trop} *are invariant.*

Tropical proof by Gathmann/M, 2007.

Tropical proof of invariance

Consider the moduli space of *marked tropical curves* and the evaluation map.



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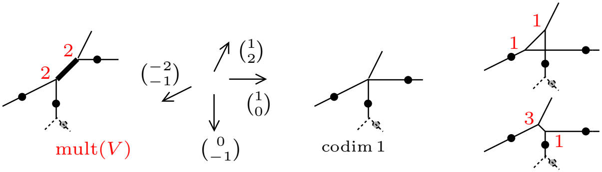
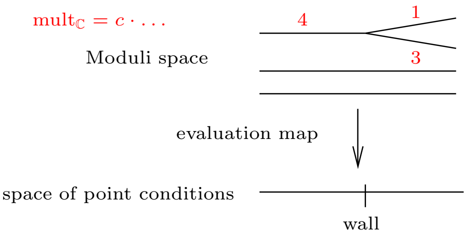




Real plane curves

A real plane curve consists

- of a defining polynomial $f \in \mathbb{R}[x, y, z]$
- and a zero-set in $\mathbb{P}^2_{\mathbb{C}}$.

Consequence:

- If $z \in \mathbb{P}^2_{\mathbb{C}}$ is in $V(f) \Rightarrow \overline{z} \in V(f)$.
- We can choose real points or pairs of complex conjugate points as conditions in a real enumerative problem.

Counting real curves

Problem:

Counting rational nodal real plane curves of degree d through r real points and s pairs of complex conjugate points satisfying $r + 2s = 3d - 1$ does not lead to an invariant number.

Example (Degtyarev, Kharlamov, 2000)

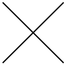
The number of real rational cubics through 8 real points is 8, 10 or 12, depending on the position of the points.

Welschinger invariants

Count real curves with a sign, depending on the nodes:

$x^2 + y^2 = 0$

solitary node



$x^2 - y^2 = 0$

Definition (Welschinger invariants)

- $\mathcal{P}$ a set of r real and s pairs of complex conjugate points
- C real plane rational curve of degree d through $\mathcal{P}$
- $m(C) := \#$ solitary nodes of C
- $W(d, r, s) := \sum_{C \text{ through } \mathcal{P}} (-1)^{m(C)}$

Welschinger invariants

Theorem (Welschinger, 2005)

The numbers $W(d, r, s)$ are invariant.

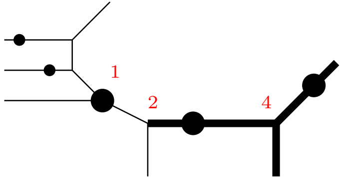
But: how can we compute them? Tropically!

Example

d	1	2	3	4
N_d	1	1	12	620
$W(d, 3d-1, 0)$	1	1	8	240

Tropical Welschinger curves

- In the tropical world, complex conjugate points correspond to special point conditions, fat points.
- A fat point has to be on a vertex, or on an even edge.
- Connected components of the even part have to meet the non-even part at one point.



$\text{mult}_{\mathbb{R}}(C) = \text{sign} \cdot \prod_V \text{mult}(V)$, where the product goes over

- vertices V with a fat point,
- vertices with an adjacent even edge.

Tropical $W(d, r, s)$

$\mathcal{P}$ a set of r thin and s fat points.

Definition

$W(d, r, s)^{\text{trop}} = \sum_{C \text{ through } \mathcal{P}} \text{mult}_{\mathbb{R}}(C)$.

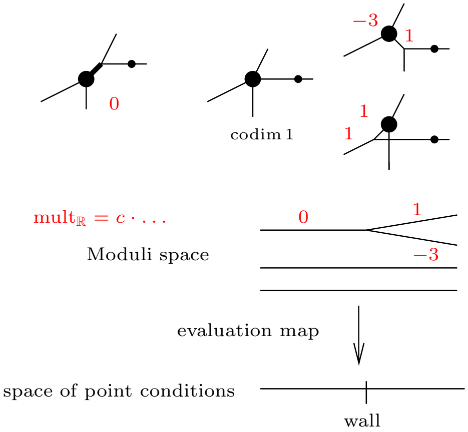
Theorem (Shustin, 2006)

$W(d, r, s) = W(d, r, s)^{\text{trop}}$.

Correspondence Thm+ Invariance of $W(d, r, s) \Rightarrow$ Invariance of $W(d, r, s)^{\text{trop}}$.

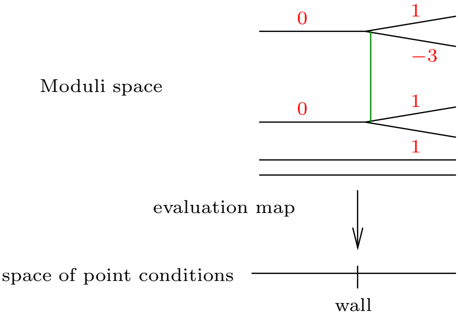
Tropical proof?

Invariance is not local

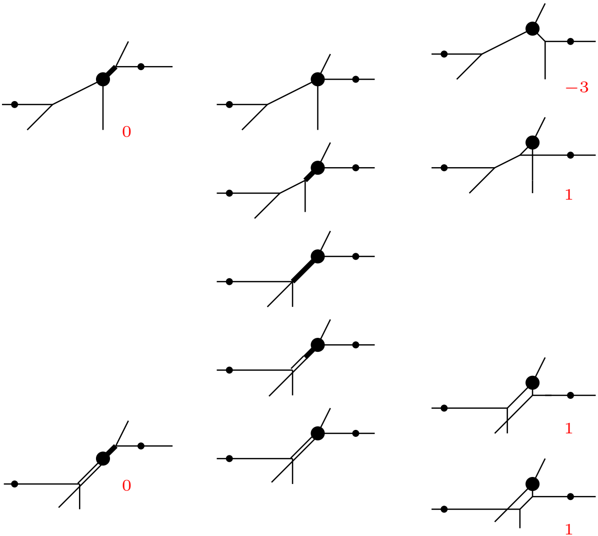


The invariance does not hold for any choice of directions for the ends.

What is happening?

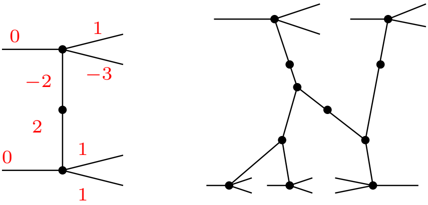


Bridges

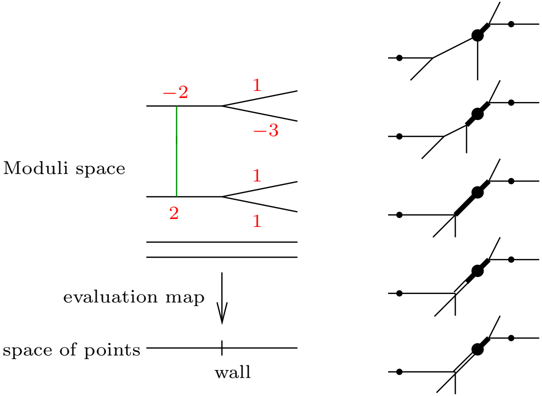


Proof of invariance

- Define bridge curves and their multiplicity.
- Bridge curves can be deformed in a 1-dimensional family by moving the string.
- “Bounds” of the 1-dimensional movement are called vertices of the bridge.
- Prove local invariance at each interior vertex of a bridge.
- Prove local invariance at each “end” of a bridge.
- Prove that every bridge has ends (here, we need degree-d-ends).



New tropical invariants



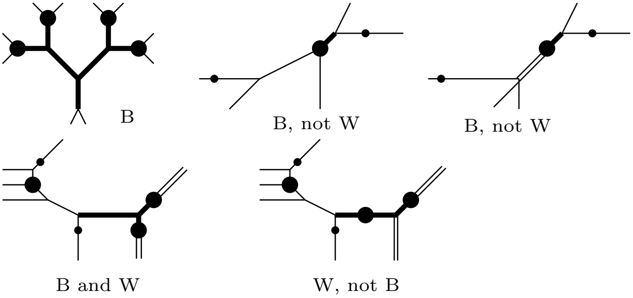
Broccoli curves

Convention: consider even ends as "double ends".

Definition

A tropical curve is a broccoli curve, if any connected component of the even part meets the non-even part in k vertices of which $k-1$ are marked by a fat point.

Careful: here, we consider double ends *not* as even.



Broccoli curves

Broccoli numbers satisfy a local invariance. Thus:

Theorem (Gathmann, Schroeter, M, 2011)

Broccoli numbers are invariant for any choice of ends.

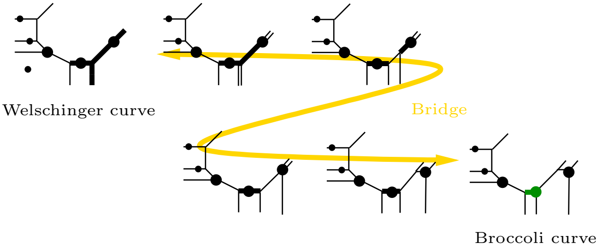
- We can generalize the definition and produce a recursive formula for broccoli curves.
- Generalization to higher genus seems possible.

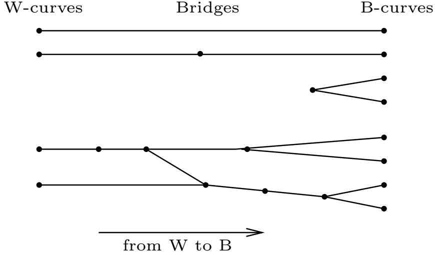
Theorem (Gathmann, Schroeter, M, 2011)

For degree-d-ends, broccoli numbers equal Welschinger numbers.

Proof by adaption of the bridge-technique.

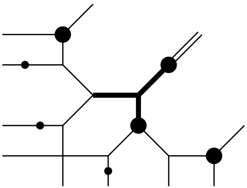
B-W-bridges


Welschinger curve
Bridge
Broccoli curve


W-curves
Bridges
B-curves
from W to B

The question

What corresponds to broccoli in the real world?



THANK YOU